Behavioral Heterogeneity and Financial Markets: 
Crossed Markets under Informationally Efficient Pricing\textsuperscript{12}

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Abstract

Consider a standard Glosten-Milgrom sequential trading model in which some traders use both their private information and historical information when making decisions, while other traders have no information and trade randomly (e.g., Glosten and Milgrom (1985), Avery and Zemsky (1998), Park and Sabourian (2013)). We introduce a third category of traders (called \textit{naïve} traders) who rely only on their private information to decide whether to trade. We find that behavioral heterogeneity provides one explanation for the existence of crossed markets (inverted bid/ask spreads) for informationally efficient prices in competitive markets.

(Preliminary draft)

1 Introduction

This paper studies the influence of behavioral heterogeneity on asset pricing in competitive financial markets. Here behavioral heterogeneity means that traders (he) have different levels of sophistication in using information when making decisions.

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Specifically, we consider a setting in the tradition of Glosten and Milgrom (1985) in which the bid and ask prices of a risky asset are set by a market maker (she) who makes zero expected profit subject to competition. As in a standard Glosten-Milgrom sequential trading model (e.g., Glosten and Milgrom (1985), Avery and Zemsky (1998), Park and Sabourian (2013)), some traders (called informed traders) use both private information and historical transaction information to derive their expected values of an asset, and some other traders (called noise traders) trade randomly. We introduce a type of traders (called naive traders). When making decisions on whether to trade and how to trade, naive traders take only their own information into account but fail to extract information from transaction history. On the other hand, the market maker knows all the historical transactions but she has no private information.

We find that behavioral heterogeneity provides one explanation for the existence of crossed markets (inverted bid/ask spreads, i.e., bid prices are greater than ask prices) for informationally efficient prices in competitive markets.

Recall that besides the information implicit in historical transactions, informed traders also have private information about the true value of the asset, while the market maker has not private information. Then compared to informed traders, the market maker has informational disadvantage, which implies that her expected profit from trading with informed traders is always negative.

In a framework without naive traders (as Avery and Zemsky (1998), Park and Sabourian (2013)), bid/ask spreads are always positive. This is because, in a competitive market without naive traders, the market maker must makes positive profit from trading with a noise trader to compensate her loss from trading with an informed trader. Notice that the market maker’s profit from selling one unit of the asset to a noise trader is the ask price minus the market value of the asset (the expected value of the asset conditional on historical information), and her profit from purchasing one unit of the asset from a noise trader is the market value of the asset minus the bid price. Then the market maker makes positive profit from trading with a noise trader if and only if the ask price is greater than the market value of the asset, and the bid price is lower than the market value of the asset. Thus the ask price is greater than the bid price, i.e., the bid/ask spread is positive.
Yet empirical data document records of crossed markets. For example, Ito and Hashimoto (2004) find that between January 1, 1999 to December 31, 2001, about 1% of the bid/ask quotes in the Yen/Dollar foreign exchange markets are inverted (the best bid prices are greater than ask prices). Shilkloha, Van Ness, and Van Ness (2008) find similar phenomena in NASDAQ and NYSE but with even greater frequencies – for an average active NYSE stock, with 4.5% of time the bid/ask spreads are inverted, and 5.79% of the sub-sample are executed. The similar statics in NASDAQ are even higher, with 10.58% of time inverted bid/ask spreads show up and 22.17% of the sub-sample are executed.

Previous literature studying the resource of crossed markets attributes their existence to informationally inefficient prices. But our paper shows that crossed markets can occur when prices are informationally efficient and markets are competitive, if traders are behaviorally heterogeneous. This is because the market maker’s gains from trading with a naive trader can exceed her loss from trading with an informed trader, then she must make negative profit from trading with a noise trader, as perfect competition requires her overall expected profit to be zero. That the market maker makes negative profit from trading with a noise trader implies that the ask price must be less than the market value of the asset, and the bid price must be greater than the market value of the asset. Thus bid price is greater than the ask price, that is, a crossed market occurs.

The introduction of behavioral heterogeneity has empirical support. Cipriani and Guarino (2005) observe some participants follow their own signals even when information cascades have occurred. Park and Sgroi (2009) observe a non-negligible proportion of the participants in their experiments stick to their signals rather than following the developments of prices, while some other traders follow the developments of prices carefully. Such “naive” actions are also observed in experiments on other topics. For example, Guarnaschelli, McKelvey, and Palfrey (2000) observe a positive proportion of participants in their experiment on collective decisions always follow their own signals.

The presence of naive traders captures several scenarios. One scenario is that naive traders are restricted from historical information or information implicit in other traders’ actions. Such restrictions can be time constraint or resource constraint. An-
other scenario is that naive traders may have low sophistication in handling information, so they fail to extract information from other traders’ actions. The third scenario is to mimic the “overconfidence” of traders in the precision of their own signals. Overconfident traders are aware of the information implicit in previous transactions, and they are capable of extracting such information correctly, but they are so confident in the precision of their own information that they act as if they always ignore external information but follow their own information only.

Several other models also investigate the possibility that some market participants have bounded rationality, but all these models explore different implications that are not the main focus of this paper, and they do not provide explanation on crossed markets.

For example, Hong and Stein (1999) model a market in which some traders only consider their own information, while other voters use very simple strategies to infer information from historical transactions. Because the second type of traders in their paper only infer very limited information from historical transactions, prices in their paper are not informationally efficient. Eyster, Rabin, and Vayanos (2013) introduce “cursed traders” into a financial market. Traders in their paper are aware of the information implicit in other traders’ behavior, but under-utilize such information. Heterogeneity in their paper comes from the differences among traders’ cursedness, that is, traders under-utilize information with different levels. Such under-utilization of information makes prices informationally inefficient.

Though our paper is not the first one examining the influence of market participants who fail to utilize information implicit in prices, to our best knowledge, this paper is the first that provides an explanation for the existence of crossed markets for informationally efficient prices and perfect competition. And the existence of crossed markets in competitive markets are consistent with Shilkloka, Van Ness, and Van Ness (2008), who find that negative bid/ask spreads show more frequently in more competitive markets.

Another implication of our paper is that in competitive financial markets with informationally efficient prices, crossed markets occur if and only if there are traders who have different types of informational disadvantages compared to the market maker. The reason is similar to the one for why crossed markets do not occur in models with
only informed traders and noise traders: the market maker always makes negative profit from trading with traders with informationally advantage, then if there is only one type of traders who have informational disadvantage, she must make positive profit from trading with these traders, which implies that ask prices are greater than the market values and bid prices are lower than the market values. If there is multiple types of traders with informational disadvantage, the market maker can make positive profit from trading with one type of these traders, and make negative profit from trading with another type of these traders who have informational disadvantage. Only in this case crossed market can occur.

This paper is organized as follows. Section 2 is on the basic setup of the model. Section 3 characterizes the sufficient and necessary conditions for the existence of crossed markets. Section 4 concludes and discuss potential extensions.

2 The Model

Security: V = \{V_1, V_2, V_3\} with V_1 < V_2 < V_3. We assume that \{V_1, V_2, V_3\} = \{0, V, 2V\}, V > 0, and that the prior distribution is symmetric around V_2; thus Pr(V_1) = Pr(V_3).

Public History \(H_t = \{(a_1, p_1), \ldots, (a_{t-1}, p_{t-1})\}\).

Market Maker is subject to competition and thus makes zero expected profits. In every period t, prior to the arrival of a trader, he posts a bid price \(bid_t\) and an ask price \(ask_t\).

Traders: There is a pool of traders that consists of three kinds of agents: institutional traders, retailer traders and noise trader. \(Pr(Informed) = \mu > 0\), \(Pr(Naive) = \theta > 0\), and \(Pr(Noise) = 1 - \mu - \theta > 0\).

Each agent receives a private, conditionally independent and identically distributed (i.i.d.) signal about the true value of the asset V. The set of possible signals is denoted by \(S\) and consists of three elements \(S_1, S_2, \) and \(S_3\), that is \(S = \{S_1, S_2, S_3\}\). \(E[V|S_1] < E[V|S_2] < E[V|S_3]\). \(Pr(S_i|V_j)\) is the probability of signal \(S_i\) if the true value of the asset is \(V_j\).

The informed traders are risk neutral and rational, by which we mean they choose the actions that maximize their expected payoff conditioning on their own informa-
tion and the information implicit in historical transactions. More precisely, an informed trader who enters the market at period $t$ and receives signal $S_t$ buys only if $E[V|H^t, S_t] - ask^t > \max \{bid^t - E[V|H^t, S_t], 0\}$, sells only if $bid^t - E[V|H^t, S_t] > \max \{E[V|H^t, S_t] - ask^t, 0\}$, and he buys or sells with equal probability if $E[V|H^t, S_t] - ask^t = bid^t - E[V|H^t, S_t] > 0$. Otherwise he holds.\(^5\)

The naive traders sell if they receive signal $S_1$, hold if they receive signal $S_2$, buy if they receive signal $S_3$. So a naive trader with a given signal $S$ sticks to the same action, regardless the relationship between $E[V|S]$ and $ask^t$, and the relationship between $E[V|S]$ and $bid^t$. In other words, a naive trader’s action does not change with the current prices. This the key difference between naive traders and the “newswatchers” in Hong and Stein (1999). In Hong and Stein (1999), a newswatcher buys if $E[V|S] > ask^t$ and sells if $E[V|S] < bid^t$, that is, his action changes if the prices changes.

Replacing naive traders with newswatchers seems to be appealing, as newswatchers’s actions seem to be more intuitive and reasonable. We did think about replacing naive traders with newswatchers, but we find that such a change does not add to our findings but only makes the analysis much more complicated. The main results in our paper come from the fact that there are multiple types of traders who have disadvantages in information. Since the bid prices and ask prices are informationally efficient, the actions adopted by traders who do not utilize all the information in the market, by which we mean both the history and private signals, are not informationally efficient. Such an under-utilization of information brings naive traders and noise traders a disadvantage in trading such disadvantages in trading that the market maker can make strictly positive profit from trading with them. But because naive traders and noise traders have different disadvantages as they utilize information differently, the market maker may gain from trading with one type of naive traders and noise traders, while lose from trading with the other type. If the market maker makes so much profit from trading with a naive trader that such profit exceeds her loss from trading with an informed trader, she must make negative profit from trading with a noise trader, and

\(^5\)The traditional assumption on informed traders’ strategies is that an informed buys only if $E[V|H^t, S_t] > ask^t$, sells only if $bid^t > E[V|H^t, S_t]$, and otherwise they hold. We do not adopt this assumption it does not work for crossed markets. Consider a crossed market with $bid^t > ask^t$, an informed trader with $E[V|H^t, S_t] \in (ask^t, bid^t)$ makes strictly positive when he sells and he buys, then his best response cannot be characterized under the traditional assumption.
then we may see a crossed market. See Section 3 for more detailed discussions.

In other word, what matters is that there is heterogeneity among the informationally inefficient actions of traders, but not the exact informationally inefficient actions. Thus replacing naive traders with newswatchers does not affect our main findings, we decide to adopt the assumption of naive traders rather than the one of newswatchers, since the latter complicates the calculation a lot.

The noise trader trade randomly. \( \gamma = \frac{1-\mu-\theta}{3} \).

Public Belief and Public Expectation: \( q_t = \Pr(V_t|H^t) \). \( \beta_t = \Pr(\text{buy}|H^t, V_t) \).

The market makers Price-Setting: \( \text{ask}^t = \mathbb{E}[V|\text{a}^t=\text{buy at ask}^t, H^t] \) and \( \text{bid}^t = \mathbb{E}[V|\text{a}^t=\text{sell at bid}^t, H^t] \). Nash Equilibrium.

The market maker lose profits on informed traders. Then, he must make profits on naive or noisy traders.

Informative Private Signals: The private signals of the informed traders are informative at history \( H^t \) if there exists \( S \in \mathcal{S} \) such that \( \mathbb{E}[V|H^t, S] \neq \mathbb{E}[V|H^t] \).


Call signal \( S_2 \) a neutral signal if \( \Pr(S_2|V_1) = \Pr(S_2|V_2) = \Pr(S_2|V_3) \). \( S_1 \) and \( S_3 \) cannot be neutral signals because they have non-zero bias. Later we show that a neutral signal is always non-informative for any history.

**Definition 1** Herding and Contrarianism\(^6\)

(D.1.1) **Herding.** A trader with signal \( S \in \mathcal{S} \) buy herds in period \( t \) at history \( H^t \) if and only if (i) \( \mathbb{E}[V|S] < \text{bid}^t \), (ii) \( \mathbb{E}[V|H^t, S] > \text{ask}^t \), or (iii-h) \( \mathbb{E}[V|H^t] > \mathbb{E}[V] \). Sell herding at history \( H^t \) is defined analogously with the required conditions \( \mathbb{E}[V|S] > \text{ask}^t \), \( \mathbb{E}[V|H^t, S] < \text{bid}^t \), and \( \mathbb{E}[V|H^t] < \mathbb{E}[V] \). Type S herds if he either buy herds or sell herds at some history.

(D.1.2) **Contrarianism.** A trader with signal \( S \) engages in buy contrarianism in period \( t \) at history \( H^t \) if and only if (i) \( \mathbb{E}[V|S] < \text{bid}^t \), (ii) \( \mathbb{E}[V|H^t, S] > \text{ask}^t \), or (iii-c) \( \mathbb{E}[V|H^t] < \mathbb{E}[V] \). Sell contrarianism at history \( H^t \) is defined analogously with the required conditions \( \mathbb{E}[V|S] > \text{ask}^t \), \( \mathbb{E}[V|H^t, S] < \text{bid}^t \), and \( \mathbb{E}[V|H^t] > \mathbb{E}[V] \).

\(^6\)We follow the definitions in Avery and Zemsky (1998) and Park and Sabourian (2011).
Type S engages in contrarianism if he engages in either buy contrarianism or sell contrarianism at some history.

3 Crossed Markets

Theorem 1 If $\mu > \theta$, for any $H^t$, $ask^t > E[V|H^t] > bid^t$.

Proof. (Theorem 1) Here we ignore the trivial case $E[V|H^t, S_1] = E[V|H^t, S_2] = E[V|H^t, S_3]$ in which $ask^t = E[V|H^t] = bid^t$ and then no informed traders trader with the market maker. We focus on the case the there exist $S, \hat{S} \in \mathcal{S}$ such that $E[V|H^t, S] > E[V|H^t, \hat{S}]$, i.e., the case in which some informed trader sells to the market maker and some other informed trader buys from the market maker.

Suppose $bid^t \geq E[V|H^t]$, then the market maker makes non-positive profit from buying from a noise trader. Since an informed trader trades with the market makers, the market maker must make strictly negative profit from buying from an informed trader. Thus the market maker must make strictly positive profit from buying from a naive trader, which implies that $bid^t > E[V|H^t, S_1]$. Thus an informed trader who receives signal $S_1$ does not sell. Moreover, since the market maker makes non-positive profit from buying from a noise trader, her total expected profit from buying from an informed trader or a naive trader is positive.

Assume $S'$ and $S''$ are other two signals, and $S'$ sell. If $S''$ sells as well, the total expected profit of the market maker makes from buying one unit from a naive trader or an informed trader is

$$
\begin{align*}
\mu Pr(S'|H^t) \{E[V|H^t, S'] - bid^t\} + \mu Pr(S''|H^t) \{E[V|H^t, S''] - bid^t\} \\
+ \theta Pr(S_1|H^t) \{E[V|H^t, S_1] - bid^t\} \\
< \mu Pr(S'|H^t) \{E[V|H^t, S'] - bid^t\} + \mu Pr(S''|H^t) \{E[V|H^t, S''] - bid^t\} \\
+ \mu Pr(S_1|H^t) \{E[V|H^t, S_1] - bid^t\} \\
= \mu \{E[V|H^t] - bid^t\} \\
\leq 0,
\end{align*}
$$

contradicting with that the market maker’ total expected profit from buying one unit from a naive trader or an informed trader is positive.
If \( S'' \) does not sell, that is \( E[V|H^t] < E[V|H^t, S''] \), the total expected profit of the market maker from buying from an informed trader or a naive trader is

\[
\mu \Pr(S'|H^t) \{E[V|H^t, S'] - \text{bid}^d\} + \theta \Pr(S_1|H^t) \{E[V|H^t, S_1] - \text{bid}^d\}
\]

which contradicts with that the market maker’s total expected profit from buying one unit from a naive trader or an informed trader is positive.

Therefore, \( \text{bid}^d \) must be strictly less than \( E[V|H^t] \).

Similarly, we have \( \text{ask}^i > E[V|H^t] \). □

**Theorem 2** Suppose that \( \mu < \theta \), \( S_2 \) is a neutral signal, \( S_1 \) is nU-shaped, and \( S_3 \) is phill-shaped. Then inverted bid/ask spreads exist, i.e., crossed markets exist.

**Proof. (Theorem 2)** This results comes straightforward from Theorem 3 and Proposition 2.

Theorem 3 shows the existence of a history with \( E[V|H^t, S_1] > E[V|H^t, S_2] > E[V|H^t, S_3] \) under the given conditions. Proposition 2 shows that at that period, \( \text{ask}^i < \text{bid}^d \), that is, inverted bid/ask spreads exist. □

**Proposition 1** Suppose that \( \mu < \theta \), \( S_2 \) is a neutral signal, \( S_1 \) is nU-shaped, and \( S_3 \) is phill-shaped. Then if \( E[V|H^t, S_3] > E[V|H^t, S_2] > E[V|H^t, S_1] \), we have \( \text{ask}^i > E[V|H^t] > \text{bid}^d \). Then an informed trader makes a sale when he receives signal \( S_1 \), holds when he receives signal \( S_2 \), and makes a purchase when he receives signal \( S_3 \).

**Proof. (Proposition 1)** By Proposition 3, we know that \( \text{bid}^d > E[V|H^t, S_1] \) and \( \text{ask}^i < E[V|H^t, S_3] \) when \( E[V|H^t, S_3] > E[V|H^t, S_2] > E[V|H^t, S_1] \). Recall that \( S_2 \) is neutral implies that \( E[V|H^t, S_2] = E[V|H^t], \forall H^t \).

\( \text{bid}^d > E[V|H^t, S_1] \) implies that the market maker receives strictly negative expected profit from buying one unit of the asset from a naive trader. Since she makes strictly
negative expected profit from selling one unit of the asset to an informed trader, her expected profit from selling one unit of the asset to a noise trader, \( \frac{1-\mu-\theta}{3} \{ E[V|H^t] - bid^t \} \), must be strictly positive, which implies that \( bid^t < E[V|H^t] \).

Similarly, the market maker must makes strictly positive profit from buying one unit of the asset from a noise trader, that is, \( \frac{1-\mu-\theta}{3} \{ ask^t - E[V|H^t] \} > 0 \).

Thus \( ask^t > E[V|H^t] \). ■

**Theorem 3** Suppose that \( \mu < \theta \), \( S_1 \) is nU-shaped, \( S_2 \) is neutral, and \( S_3 \) is phill-shaped. Then there exists some history \( H^t \) such that \( E[V|H^t, S_1] > E[V|H^t] > E[V|H^t, S_3] \).

**Proof.** (Theorem 3) Consider the history \( H^\infty \) with \( a^t = buy, \forall t \). Suppose that an informed trader who receives signal \( S_1 \) never buys. Then we must have \( E[V|H^t, S_1] \leq E[V|H^t] \).

To see this clear, recall that \( E[V|H^t, S_2] = E[V|H^t] \), then at any \( t \), if \( E[V|H^t, S_1] > E[V|H^t] \), we must have \( E[V|H^t, S_3] < E[V|H^t] \). Thus by Proposition 3, \( E[V|H^t, S_1] > ask^t \), which implies that an informed trader buys from the market maker, conflicting with the assumption that an informed trader who receives signal \( S_1 \) does not buy at any period of this history.

Then at each period \( t \), we have (i) \( E[V|H^t, S_3] > E[V|H^t] = E[V|H^t, S_2] > E[V|H^t, S_1] \), or (ii) \( E[V|H^t, S_3] = E[V|H^t] = E[V|H^t, S_2] = E[V|H^t, S_1] \). In both cases, \( E[V|H^t, S_3] > E[V|H^t] \), then for any \( t \),

\[
q_1^t q_2^t [Pr(S_3|V_2) - Pr(S_3|V_1)] + q_2^t q_3^t [Pr(S_3|V_3) - Pr(S_3|V_2)] + 2q_1^t q_3^t [Pr(S_3|V_3) - Pr(S_3|V_1)] \geq 0,
\]

which is equivalent to

\[
\frac{q_1^t}{q_3^t} [Pr(S_3|V_2) - Pr(S_3|V_1)] + [Pr(S_3|V_3) - Pr(S_3|V_2)] + \frac{2q_1^t}{q_2^t} [Pr(S_3|V_3) - Pr(S_3|V_1)] \geq 0.
\]

Consider any period \( t \) with (i) first. At this period, an informed trader buys from the market maker only if he receives signal \( S_3 \), then

\[
\beta_i^t = Pr(buy|H^t, V_i)
= \mu Pr(S_3|H^t, V_i) + \theta Pr(S_3|H^t, V_i) + \frac{1-\mu-\theta}{3}
= (\mu+\theta) Pr(S_3|V_i) + \frac{1-\mu-\theta}{3}.
\]

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At any period $t$ with (ii), no informed traders trade with the market maker, then
\[ \beta_t^i = \Pr(bay|H^t, V_i) = \theta \Pr(S_3|H^t, V_i) + \frac{1 - \mu - \theta}{3} = \theta \Pr(S_3|V_i) + \frac{1 - \mu - \theta}{3}. \]

Since $S_3$ is pill-shaped, we have
\[
1 > \frac{\theta \Pr(S_3|V_i) + \frac{1 - \mu - \theta}{3}}{\theta \Pr(S_3|V_i) + \frac{1 - \mu - \theta}{3}} \geq \beta_t^i, \quad i = 2, 3
\]
which implies that for any $t$,
\[
1 > \left[ \frac{\theta \Pr(S_3|V_i) + \frac{1 - \mu - \theta}{3}}{\theta \Pr(S_3|V_i) + \frac{1 - \mu - \theta}{3}} \right]^t \geq \left[ \frac{(\mu + \theta) \Pr(S_3|V_i) + \frac{1 - \mu - \theta}{3}}{(\mu + \theta) \Pr(S_3|V_i) + \frac{1 - \mu - \theta}{3}} \right]^t, \quad i = 2, 3.
\]
As $q_3^i > q_3^i$ in both cases, we know that at any period $t$ of this history, $E[V^i|H^t] > E[V]$. Since $\Pr(S_2|V_2) - \Pr(S_3|V_1) > 0$ and $\Pr(S_3|V_3) - \Pr(S_3|V_1) > 0$,
\[ \frac{q_3^i}{q_3^i} [\Pr(S_3|V_2) - \Pr(S_3|V_1)] + [\Pr(S_3|V_3) - \Pr(S_3|V_2)] + 2 \frac{q_3^i}{q_2^i} [\Pr(S_3|V_3) - \Pr(S_3|V_1)] \]
\[ \geq \left[ \frac{(\mu + \theta) \Pr(S_3|V_1) + \frac{1 - \mu - \theta}{3}}{(\mu + \theta) \Pr(S_3|V_1) + \frac{1 - \mu - \theta}{3}} \right]^t [\Pr(S_3|V_2) - \Pr(S_3|V_1)] + [\Pr(S_3|V_3) - \Pr(S_3|V_2)] + \frac{q_3^i}{q_3^i} [\Pr(S_3|V_3) - \Pr(S_3|V_1)] \]
and
\[ \frac{q_3^i}{q_3^i} [\Pr(S_3|V_2) - \Pr(S_3|V_1)] + [\Pr(S_3|V_3) - \Pr(S_3|V_2)] + 2 \frac{q_3^i}{q_2^i} [\Pr(S_3|V_3) - \Pr(S_3|V_1)] \]
\[ \leq \left[ \frac{\theta \Pr(S_3|V_1) + \frac{1 - \mu - \theta}{3}}{\theta \Pr(S_3|V_1) + \frac{1 - \mu - \theta}{3}} \right]^t [\Pr(S_3|V_2) - \Pr(S_3|V_1)] + [\Pr(S_3|V_3) - \Pr(S_3|V_2)] + \frac{q_3^i}{q_3^i} [\Pr(S_3|V_3) - \Pr(S_3|V_1)] . \]
As both
\[ \left[ \frac{(\mu + \theta) \Pr(S_3|V_1) + \frac{1 - \mu - \theta}{3}}{(\mu + \theta) \Pr(S_3|V_1) + \frac{1 - \mu - \theta}{3}} \right]^t [\Pr(S_3|V_2) - \Pr(S_3|V_1)] + [\Pr(S_3|V_3) - \Pr(S_3|V_2)] + \frac{q_3^i}{q_3^i} [\Pr(S_3|V_3) - \Pr(S_3|V_1)] \]
and
\[ \left[ \frac{\theta \Pr(S_3|V_1) + \frac{1 - \mu - \theta}{3}}{\theta \Pr(S_3|V_1) + \frac{1 - \mu - \theta}{3}} \right]^t [\Pr(S_3|V_2) - \Pr(S_3|V_1)] + [\Pr(S_3|V_3) - \Pr(S_3|V_2)] + \frac{q_3^i}{q_3^i} [\Pr(S_3|V_3) - \Pr(S_3|V_1)] . \]
converge to $\Pr(S_3|V_3) - \Pr(S_3|V_2) < 0$ when $t$ increases to infinity, there exists some
$t' > 1$ such that\footnote{\label{footnote1} $E[V|S_3] > E[V|S_2] > E[V|S_1]$ implies that $E[V|H^1, S_3] > ask^1$}

\[
\frac{q_1^t}{q_3^t} [Pr(S_3|V_2) - Pr(S_4|V_1)] + [Pr(S_4|V_3) - Pr(S_3|V_2)] + 2 \frac{q_1^t}{q_2^t} [Pr(S_3|V_3) - Pr(S_4|V_1)] < 0,
\]

which implies that $E[V|H^t', S_3] < E[V|H^t]$. Notice that once $E[V|H^t', S_3] < E[V|H^t]$, we have $E[V|H^t', S_1] > E[V|H^t]$. Then by Proposition 1, we have $E[V|H^t', S_1] > ask^t$ and $E[V|H^t', S_3] < bid^t$, which implies that at period $t'$, an informed trader buys if he receives signal $S_1$ and sells if he receives signal $S_3$. This contradicts with the assumption that an informed trader with signal $S_1$ never buys.

Thus at the given history $H^\infty$, an informed trader who enters the market at period $t'$ buys from the market maker when he receives signal $S_1$ and sells to the market maker when he receives signal $S_3$. \]

\textbf{Proposition 2} Suppose that $\mu < \theta$, $S_2$ is a neutral signal, $S_1$ is \textit{nU}-shaped, and $S_3$ is \textit{phill}-shaped. Then if $E[V|H^t', S_1] > E[V|H^t', S_2] > E[V|H^t', S_3]$, we have $bid^t > ask^t$.

\textbf{Proof. (Proposition 2)} We prove this result by showing that $bid^t > E[V|H^t]$ and $ask^t < E[V|H^t]$.

Suppose that in an equilibrium under the given conditions, we have $bid^t \leq E[V|H^t]$. Then the only informed trader who sells to the market maker is an informed trader with signal $S_3$. As the market maker makes 0 profit from buying one nit of the asset from a trader, we have

\[
\frac{\mu Pr(S_3|H^t)}{E[V|H^t, S_3] - bid^t} + \theta Pr(S_1|H^t) \{E[V|H^t, S_1] - bid^t\} + \frac{1 - \mu - \theta}{3} \{E[V|H^t] - bid^t\} = 0.
\]

By $E[V|H^t', S_1] > E[V|H^t'] \geq bid^t$, we know that

\[
\mu Pr(S_3|H^t) \{E[V|H^t, S_3] - bid^t\} + \mu Pr(S_1|H^t) \{E[V|H^t, S_1] - bid^t\} < \frac{\mu Pr(S_3|H^t)}{E[V|H^t, S_3] - bid^t} + \theta Pr(S_1|H^t) \{E[V|H^t, S_1] - bid^t\} \leq 0,
\]

then $Pr(S_3|H^t)E[V|H^t, S_3] + Pr(S_1|H^t) E[V|H^t, S_1] < Pr(S_3|H^t) bid^t + Pr(S_1|H^t) bid^t$, which is equivalent to $E[V|H^t'] - Pr(S_2|H^t) E[V|H^t, S_2] = [1 - Pr(S_2|H^t)] E[V|H^t] < [1 - Pr(S_2|H^t)] bid^t$, contradicting with $E[V|H^t] \geq bid^t$. Thus $bid^t > E[V|H^t]$. \footnote{\ref{footnote1}}
Similarly, suppose that \( \text{ask}^t \geq E[V|H^t] \), then an informed trader buys from the market maker only if he receives signal \( S_1 \). By the 0-profit assumption, we have

\[
\mu Pr(S_1|H^t) \{ \text{ask}^t - E[V|H^t, S_1] \} + \theta Pr(S_3|H^t) \{ \text{ask}^t - E[V|H^t, S_3] \} + \frac{1 - \mu - \theta}{3} \{ \text{ask}^t - E[V|H^t] \} = 0.
\]

By \( \text{ask}^t \geq E[V|H^t] > E[V|H^t, S_3] \), we know that

\[
\mu Pr(S_1|H^t) \{ \text{ask}^t - E[V|H^t, S_1] \} + \mu Pr(S_3|H^t) \{ \text{ask}^t - E[V|H^t, S_3] \} < 0,
\]

then \( Pr(S_1|H^t)E[V|H^t, S_1] + Pr(S_3|H^t)E[V|H^t, S_3] > Pr(S_1|H^t)\text{ask}^t + Pr(S_3|H^t)\text{ask}^t \), which is equivalent to \( E[V|H^t] - Pr(S_2|H^t)E[V|H^t, S_2] = [1 - Pr(S_2|H^t)] E[V|H^t] + [1 - Pr(S_2|H^t)] \text{ask}^t \), contradicting with \( E[V|H^t] \leq \text{ask}^t \). Thus \( \text{ask}^t < E[V|H^t] \).

Thus we have \( \text{bid}^t > \text{ask}^t \). □

Recall that only informed traders and naive traders take actions that are related to their own signals, while noise traders’ actions reveal no information. In other words, only informed traders and naive traders bring information into the market. Let \( \xi = \theta + \mu \in (0, 1) \). Then \( \xi \) is the probability that a trader brings information into the market. Call it information ratio. Since \( \theta > \mu \), we have \( \mu < \frac{1}{2}\xi \).

**Theorem 4** If \( S_1 \) is nU-shaped, \( S_2 \) is decreasing, and \( S_3 \) is phill-shaped, then for any \( \xi \in (0, 1) \), there exists \( \overline{\mu} \in (0, \frac{1}{2}\xi) \) such that for any \( \mu < \overline{\mu} \), a crossed market occurs.

**Proof. (Theorem 4)** Consider a history \( H^\infty \) such that at each period \( t \), \( a^t = \text{buy} \). Suppose that informed traders with signal \( S_1 \) never buy on this history. Recall that \( S_2 \) is decreasing implies that for any given history \( H^t \), \( E[V|H^t, S_2] < E[V|H^t] \).

**Claim T.4.1:** \( E[V|H^t, S_2] \geq \text{ask}^t \) only if \( E[V|H^t, S_1] > \text{ask}^t \).

Suppose that at some history \( H^t \), we have \( E[V|H^t, S_2] \geq \text{ask}^t \) and \( E[V|H^t, S_3] \geq \text{ask}^t \). \( E[V|H^t, S_3] \geq \text{ask}^t \) implies that the market maker makes non-positive profit from selling one unit of the asset to a naive trader. Since the market maker always makes negative profit from trading with an informed trader, she must make strictly positive profit from selling one unit of the asset to a noise trader, which requires \( \text{ask}^t > E[V|H^t] \).
Then \( E[V|H^t, S_2] \geq \text{ask}^t > E[V|H^t] \), which is a contradiction since \( S_2 \) is decreasing. Thus \( E[V|H^t, S_3] < \text{ask}^t \).

Suppose that \( E[V|H^t, S_1] \leq \text{ask}^t \), then together with \( E[V|H^t, S_3] < \text{ask}^t \), we know that \( E[V|H^t, S_2] > E[V|H^t] \), which is a contradiction. Thus \( E[V|H^t, S_1] > \text{ask}^t \).

**Claim T.4.2:** If \( E[V|H^t, S] \geq \text{ask}^t, \forall S \in \{S_1, S_2\} \), then \( E[V|H^t, S_1] > E[V|H^t] \).

By Proposition 3, we know that given any history and any period, there always exists a signal \( S' \in \{S_1, S_2, S_3\} \) s.t. \( E[V|H^t, S'] < \text{ask}^t \). Thus if \( E[V|H^t, S_1] \geq \text{ask}^t \) and \( E[V|H^t, S_2] \geq \text{ask}^t \), we must have \( E[V|H^t, S_3] < \text{ask}^t \). Then \( E[V|H^t, S_1] \leq E[V|H^t] \) implies that \( E[V|H^t, S_2] > E[V|H^t] \), contradicting with \( E[V|H^t, S_2] < E[V|H^t] \).

**Claim T.4.3:** If \( E[V|H^t, S_1] \geq \text{ask}^t \) and \( E[V|H^t, S_3] \geq \text{ask}^t \), then \( E[V|H^t, S_1] > E[V|H^t] \).

Since an informed trader buys when he receives signal \( S_1 \) and \( S_3 \) buys at this period, we know that the market maker receives strictly negative expected profit from selling one unit of the asset to an informed trader or a naive trader. Then the market maker must make strictly positive profit from selling one unit of the asset to a noise trader, i.e., \( \text{ask}^t - E[V|H^t] > 0 \). Thus \( E[V|H^t, S_1] \geq \text{ask}^t \geq E[V|H^t] \).

**Claim T.4.4:** There exists \( t_{\xi 1} \) and \( t_{\xi 3} \) such that \( E[V|H^{t_{\xi 1}}, S_1] > E[V|H^t] \) and \( E[V|H^{t_{\xi 3}}, S_3] < E[V|H^t] \).

Suppose at any period of this history, \( E[V|H^t, S_1] \leq E[V|H^t] \). By Claim 4.1 and Claim 4.2, we know that \( E[V|H^t, S_2] < \text{ask}^t \). Since \( E[V|H^t, S_1] \leq E[V|H^t, S] \) and \( E[V|H^t, S_2] < E[V|H^t] \), we have \( E[V|H^t, S_3] > E[V|H^t] > E[V|H^t, S], \forall S \in \{S_1, S_2\} \), which implies that \( E[V|H^t, S_3] > \text{ask}^t \). Then by Claim 4.3, we have that an informed trader does not buy at this period when he receives signal \( S_1 \), which implies that all the buys in the previous history are made or by informed traders with signal \( S_3 \), or by naive traders with signal \( S_3 \), or by noise traders. Then at each period \( t \), \( \beta_i^t = (\theta + \mu) \Pr(S_3|V_i) + \frac{1-\theta-\mu}{3} \), which implies that \( \frac{q_4^t}{q_4^t} = \frac{\Pr(H^t|V_i)}{\Pr(H^t|V_i) + \Pr(V_i)} \cdot \frac{\Pr(V_i)}{\Pr(V_i) + \frac{1-\theta-\mu}{3}} \cdot \frac{t-1}{\Pr(V_3)}, \forall i = 2, 3 \).

As \( S_3 \) is phill-shaped, \( \frac{(\theta + \mu)\Pr(S_3|V_i) + \frac{1-\theta-\mu}{3}}{(\theta + \mu)\Pr(S_3|V_i) + \frac{1-\theta-\mu}{3}} < 1 \), then \( \lim_{t \to \infty} \frac{q_4^t}{q_4^t} = 0 \).

Recall that for each signal \( S \), \( E[V|H^t, S] - E[V|H^t] \) has the same sign as

\[
q_{\xi 1}^t q_{\xi 2}^t \Pr(S|V_2) - \Pr(S|V_1) + q_{\xi 1}^t q_{\xi 3}^t \Pr(S|V_3) - \Pr(S|V_2) + 2q_{\xi 1}^t q_{\xi 3}^t \Pr(S|V_3) - \Pr(S|V_1)],
\]
which is equivalent to
\[
\frac{q_1'}{q_3'} [Pr(S|V_2) - Pr(S|V_1)] + [Pr(S|V_3) - Pr(S|V_2)] + 2 \frac{q_1'}{q_2'} [Pr(S|V_3) - Pr(S|V_1)].
\]

Then
\[
\lim_{t \to \infty} \frac{q_1'}{q_3'} [Pr(S_1|V_2) - Pr(S_1|V_1)] + [Pr(S_1|V_3) - Pr(S_1|V_2)] + 2 \frac{q_1'}{q_2'} [Pr(S_1|V_3) - Pr(S_1|V_1)]
\]
\[
= Pr(S_1|V_3) - Pr(S_1|V_2) > 0,
\]

which implies that there exists some \(t'\) s.t. at any period \(t > t'\),
\[
\frac{q_1'}{q_3'} [Pr(S_1|V_2) - Pr(S_1|V_1)] + [Pr(S_1|V_3) - Pr(S_1|V_2)] + 2 \frac{q_1'}{q_2'} [Pr(S_1|V_3) - Pr(S_1|V_1)] > 0,
\]

which implies that \(E[V|H', S_1] > E[V|H']\), contradicting with the assumption that \(E[V|H', S_1] < E[V|H']\) at every period. Thus there exists some periods of this history at which \(E[V|H', S_1] \leq E[V|H']\). Let \(t_{\xi 1}\) denote the earliest one of such periods. That is, at each period \(t < t_{\xi 1}\), we have \(E[V|H', S_1] \leq E[V|H']\).

Similarly, we know that there exists some period \(t''\) such that
\[
\frac{q_1'}{q_3'} [Pr(S_3|V_2) - Pr(S_3|V_1)] + [Pr(S_3|V_3) - Pr(S_3|V_2)] + 2 \frac{q_1'}{q_2'} [Pr(S_3|V_3) - Pr(S_3|V_1)] < 0
\]
for any period \(t > t''\), which implies that \(E[V|H', S_3] < E[V|H']\) at this period, contradicting with \(E[V|H', S_3] > E[V|H']\). Thus there exists some periods of this given history at which \(E[V|H', S_3] < E[V|H']\). Let \(t_{\xi 3}\) denote the earliest one of such periods, then \(E[V|H', S_3] \geq E[V|H']\) at any \(t < t_{\xi 3}\).

Notice that Claim 4.4 implies that given other parameters unchanged, \(t_{\xi 1}\) and \(t_{\xi 3}\) depend on the value of \(\xi\) only, and they do not change if the the ratio between \(\mu\) and \(\theta\) change. Thus given \(\xi\) and other parameters unchanged, \(E[V|H'^{t_{\xi 1}}]\), \(E[V|H'^{t_{\xi 3}}]\), \(E[V|H'^{t_{\xi 1}}, S]\), \(E[V|H'^{t_{\xi 3}}, S]\), \(Pr(S|H'^{t_{\xi 1}})\), and \(Pr(S|H'^{t_{\xi 3}})\), \(\forall S \in \{S_1, S_2, S_3\}\), are constant, regardless of the ratio between \(\mu\) and \(\theta\).

Claim T.4.5: \(t_{\xi 1} \leq t_{\xi 3}\).

Suppose \(t_{\xi 1} > t_{\xi 3}\), then \(E[V|H'^{t_{\xi 3}}, S_3] < E[V|H'^{t_{\xi 1}}]\) and \(E[V|H'^{t_{\xi 3}}, S_1] \leq E[V|H'^{t_{\xi 1}}]\). This cannot hold because \(E[V|H'^{t_{\xi 3}}, S_2] < E[V|H'^{t_{\xi 1}}]\). Thus \(t_{\xi 1} \leq t_{\xi 3}\).

Claim T.4.6: There exists \(\overline{\mu} \in (0, \xi)\) s.t. for any given \(\mu < \overline{\mu}\), there exists a history
at which \( \text{bid}^{t_{\xi_1}} > \text{E}[V|H^{t_{\xi_1}}] \) and \( \text{ask}^{t_{\xi_1}} > \text{E}[V|H^{t_{\xi_1}}] \), or \( \text{bid}^{t_{\xi_1}} > \text{E}[V|H^{t_{\xi_1}}] > \text{ask}^{t_{\xi_1}} \).

Consider any given \( \xi \in (0,1) \). Let \( \text{ask}^{t_{\xi_1}}_{\xi,\mu} \) and \( \text{bid}^{t_{\xi_1}}_{\xi,\mu} \) denote the ask price and bid price at period \( t_{\xi_1} \) for a given \( \mu \), \( \text{E}\pi_{s}^{t_{\xi_1}}(\mu,\xi) \) denote the expected profit the market maker receives when selling a unit of the asset to a trader, and \( \text{E}\pi_{b}^{t_{\xi_1}}(\mu,\xi) \) denote the expected profit she receive from buying a unit of the asset from a trader.

**Case T.4.6.1:** \( t_{\xi_1} < t_{\xi_3} \).

At period \( t_{\xi_1} \), we have \( \text{E}[V|H^{t_{\xi_1}},S_1] > \text{E}[V|H^{t_{\xi_1}}] \) and \( \text{E}[V|H^{t_{\xi_1}},S_3] > \text{E}[V|H^{t_{\xi_1}}] \), then an informed trader at period \( t_{\xi_1} \) sells if he receives signal \( S_2 \).

(T.4.6.1.i) *an informed trader buys at period \( t_{\xi_1} \) if he receives signal \( S_3 \).*

In this case, \( \text{E}[V|H^{t_{\xi_1}},S_3] > \text{ask}^{t_{\xi_1}}_{\xi,\mu} \), then the market maker receives strictly negative expected profit from selling one unit of the asset to an informed trader or a naive trader. Then the market maker must make strictly positive profit from selling one unit of the asset to a noise trader, i.e., \( \text{ask}^{t_{\xi_1}}_{\xi,\mu} > \text{E}[V|H^{t_{\xi_1}}] > 0 \).

If \( \text{bid}^{t_{\xi_1}}_{\xi,\mu} \geq \text{E}[V|H^{t_{\xi_1}},S_1] \), then \( \text{bid}^{t_{\xi_1}}_{\xi,\mu} > \text{E}[V|H^{t_{\xi_1}}] \), i.e., a bubble occurs.

If \( \text{bid}^{t_{\xi_1}}_{\xi,\mu} < \text{E}[V|H^{t_{\xi_1}},S_1] \), then an informed trader sells only if he receives signal \( S_2 \). Thus the market maker’s expected profit from buying a unit of the asset from a trader is

\[
\text{E}\pi_{b}^{t_{\xi_1}}(\mu,\xi) = \mu \text{Pr}(S_2|H^{t_{\xi_1}}) \{ \text{E}[V|H^{t_{\xi_1}},S_2] - \text{bid}^{t_{\xi_1}}_{\xi,\mu} \} + \frac{1 - \mu}{3} \{ \text{E}[V|H^{t_{\xi_1}}] - \text{bid}^{t_{\xi_1}}_{\xi,\mu} \}
+ \mu \text{Pr}(S_1|H^{t_{\xi_1}}) \{ \text{E}[V|H^{t_{\xi_1}},S_1] - \text{bid}^{t_{\xi_1}}_{\xi,\mu} \} + \frac{1 - \xi}{3} \{ \text{E}[V|H^{t_{\xi_1}}] - \text{bid}^{t_{\xi_1}}_{\xi,\mu} \}
+ (\xi - \mu) \text{Pr}(S_1|H^{t_{\xi_1}}) \{ \text{E}[V|H^{t_{\xi_1}},S_1] - \text{bid}^{t_{\xi_1}}_{\xi,\mu} \},
\]

then as \( \text{E}\pi_{b}^{t_{\xi_1}}(\mu,\xi) = 0 \), we have

\[
\text{bid}^{t_{\xi_1}}_{\xi,\mu} = \frac{\mu \text{Pr}(S_2|H^{t_{\xi_1}}) \text{Pr}(S_1|H^{t_{\xi_1}}) \text{E}[V|H^{t_{\xi_1}}] + (\xi - \mu) \text{Pr}(S_1|H^{t_{\xi_1}}) \text{E}[V|H^{t_{\xi_1}}] + \frac{1 - \xi}{3} \text{E}[V|H^{t_{\xi_1}}]}{\mu \text{Pr}(S_2|H^{t_{\xi_1}}) + (\xi - \mu) \text{Pr}(S_1|H^{t_{\xi_1}}) + \frac{1 - \xi}{3}},
\]

which is decreasing in \( \mu \) when \( \xi \) is given. As for any given \( \xi \in (0,1) \),

\[
\lim_{\mu \to 0} \text{bid}^{t_{\xi_1}}_{\xi,\mu} = \frac{\text{Pr}(S_1|H^{t_{\xi_1}}) \text{E}[V|H^{t_{\xi_1}}] + \frac{1 - \xi}{3} \text{E}[V|H^{t_{\xi_1}}]}{\text{Pr}(S_1|H^{t_{\xi_1}}) + \frac{1 - \xi}{3}} > \text{E}[V|H^{t_{\xi_1}}],
\]

which implies that for the given \( \xi \), there exists \( \bar{\mu}_{3b} \in (0,\xi) \) such that for any \( \mu < \bar{\mu}_{3b} \), \( \text{bid}^{t_{\xi_1}}_{\xi,\mu} > \text{E}[V|H^{t_{\xi_1}}] \). Thus a crossed market occurs.
\((T.4.6.1.ii)\) an informed trader holds at period \(t_{\xi_1}\) if he receives signal \(S_3\).

In this case, \(\text{ask}_{\xi_1}^{S_3} \geq E[V|H^{t_{\xi_1}}, S_3] \geq \text{bid}_{\xi_1}^{S_3}\), then at this period, an informed trader buys only if he receives signal \(S_1\), and sells only if he receives signal \(S_2\).

Thus the market maker’s expected profit from buying a unit of the asset from a trader is

\[
E_{b}^{t_{\xi_1}}(\mu, \xi) = \mu \Pr(S_2|H^{t_{\xi_1}}) \left\{ E[V|H^{t_{\xi_1}}, S_2] - \text{bid}_{\xi_1}^{S_3} \right\} + \frac{1 - \theta - \mu}{3} \left\{ E[V|H^{t_{\xi_1}}] - \text{bid}_{\xi_1}^{S_3} \right\} \\
+ \theta \Pr(S_1|H^{t_{\xi_1}}) \left\{ E[V|H^{t_{\xi_1}}, S_1] - \text{bid}_{\xi_1}^{S_3} \right\} \\
= \mu \Pr(S_2|H^{t_{\xi_1}}) \left\{ E[V|H^{t_{\xi_1}}, S_2] - \text{bid}_{\xi_1}^{S_3} \right\} + \frac{1 - \xi}{3} \left\{ E[V|H^{t_{\xi_1}}] - \text{bid}_{\xi_1}^{S_3} \right\} \\
+ (\xi - \mu) \Pr(S_1|H^{t_{\xi_1}}) \left\{ E[V|H^{t_{\xi_1}}, S_1] - \text{bid}_{\xi_1}^{S_3} \right\},
\]

then as \(E_{b}^{t_{\xi_1}}(\mu, \xi) = 0\), we have

\[
\text{bid}_{\xi_1}^{S_3} = \frac{\mu \Pr(S_2|H^{t_{\xi_1}})E[V|H^{t_{\xi_1}}, S_2] + (\xi - \mu) \Pr(S_1|H^{t_{\xi_1}})E[V|H^{t_{\xi_1}}, S_1] + \frac{1 - \xi}{3} \left\{ E[V|H^{t_{\xi_1}}] - \text{bid}_{\xi_1}^{S_3} \right\}}{\mu \Pr(S_2|H^{t_{\xi_1}}) + (\xi - \mu) \Pr(S_1|H^{t_{\xi_1}}) + \frac{1 - \xi}{3}},
\]

which is decreasing in \(\mu\) when \(\xi\) is given. As for any given \(\xi \in (0, 1),\)

\[
\lim_{\mu \to 0} \text{bid}_{\xi_1}^{S_3} = \frac{\xi \Pr(S_1|H^{t_{\xi_1}})E[V|H^{t_{\xi_1}}, S_1] + \frac{1 - \xi}{3} E[V|H^{t_{\xi_1}}]}{\xi \Pr(S_1|H^{t_{\xi_1}}) + \frac{1 - \xi}{3}} > E[V|H^{t_{\xi_1}}],
\]

which implies that for the given \(\xi\), there exists \(\mu_{3b} \in (0, \xi)\) such that for any \(\mu < \mu_{3b}, \text{bid}_{\xi_1}^{S_3} > E[V|H^{t_{\xi_1}}].\)

As \(\text{ask}_{\xi_1}^{S_3} \geq E[V|H^{t_{\xi_1}}, S_3] > E[V|H^{t_{\xi_1}}]\), we know a bubble occurs.

\((T.4.6.1.iii)\) an informed trader sells at period \(t_{\xi_1}\) if he receives signal \(S_3\).

In this case, an informed trader buys from the market maker only if he receives signal \(S_1\). Suppose \(\text{ask}_{\xi_1}^{S_3} \leq E[V|H^{t_{\xi_1}}]\), then \(\text{ask}_{\xi_1}^{S_3} < E[V|H^{t_{\xi_1}}, S_1]\) and \(\text{ask}_{\xi_1}^{S_3} < E[V|H^{t_{\xi_1}}, S_3]\), which implies that the market maker makes strictly negative profit from selling one unit of the asset to an informed trader or a naive trader. Thus the market maker must make strictly positive profit from selling an unit of the asset to a noise trader, which requires \(\text{ask}_{\xi_1}^{S_3} > E[V|H^{t_{\xi_1}}]\), contradicting with \(\text{ask}_{\xi_1}^{S_3} \leq E[V|H^{t_{\xi_1}}]\). Therefore \(\text{ask}_{\xi_1}^{S_3}\) must be strictly greater than \(E[V|H^{t_{\xi_1}}]\).

That an informed trader sells when he receives signal \(S_3\) implies \(E[V|H^{t_{\xi_1}}, S_3] < \text{bid}_{\xi_1}^{S_1}\), then \(\text{bid}_{\xi_1}^{S_1} > E[V|H^{t_{\xi_1}}]\).

Given \(\xi \in (0, 1),\) let \(\mu_\xi = \min\{\mu_{3b}, \mu_{3h}, \frac{1}{2}\xi\}\), then (1.i), (1.ii), and (1.iii) imply that
if $t_{\xi} < t_{\xi}$, for any $\mu \in (0, \mu_{\xi})$, $\text{bid}^{t_{\xi}} > E[V|H^{t_{\xi}}]$ and $\text{ask}^{t_{\xi}} > E[V|H^{t_{\xi}}]$. Thus a bubble occurs.

Notice that we require $\mu < \frac{1}{2} \xi$ here, as $\mu > \frac{1}{2} \xi$ implies $\mu > \theta$, then $\text{ask}^{t_{\xi}} > E[V|H^{t_{\xi}}] > \text{bid}^{t_{\xi}}$ by Theorem 1.

**Case T.4.6.2: $t_{\xi} = t_{\xi}$.**

In this case, $E[V|H^{t_{\xi}}, S_1] > E[V|H^{t_{\xi}}] > E[V|H^{t_{\xi}}, S_3]$. Since $S_2$ is decreasing, we also have $E[V|H^{t_{\xi}}, S_2] < E[V|H^{t_{\xi}}]$. Then an informed trader buys when he receives signal $S_1$. That is, $E[V|H^{t_{\xi}}, S_1] > \text{ask}^{t_{\xi}}$. We also have $E[V|H^{t_{\xi}}, S_1] > \text{bid}^{t_{\xi}}$.

Notice that if $E[V|H^{t_{\xi}}, S_3] \geq \text{ask}^{t_{\xi}}$, the market maker receives strictly negative expected profit from selling one unit of the asset to an informed trader or a naive trader.

Then the market maker must make strictly positive profit from selling one unit of the asset to a noise trader, i.e., $\text{ask}^{t_{\xi}} > E[V|H^{t_{\xi}}]$, which implies that $\text{ask}^{t_{\xi}} > E[V|H^{t_{\xi}}, S_3]$, contradicting with $E[V|H^{t_{\xi}}, S_3] \geq \text{ask}^{t_{\xi}}$. Thus we must have $E[V|H^{t_{\xi}}, S_3] < \text{ask}^{t_{\xi}}$, i.e., an informed trader does not buy when he receives signal $S_3$.

Suppose that $E[V|H^{t_{\xi}}, S_3] \geq \text{bid}^{t_{\xi}}$. From above we have $\text{ask}^{t_{\xi}} \geq E[V|H^{t_{\xi}}, S_3] \geq \text{bid}^{t_{\xi}}$, then at this period, an informed trader buys only if he receives signal $S_1$, and sells only if he receives signal $S_2$.

Notice that $E[V|H^{t_{\xi}}, S_3] \geq \text{bid}^{t_{\xi}}$ implies $\text{bid}^{t_{\xi}} \leq E[V|H^{t_{\xi}}]$. Then the market makers’ expected profit from purchasing one unit of the asset of a trader is

$$E\pi^{t_{\xi}}_b(\mu, \xi) = \mu \Pr(S_2|H^{t_{\xi}}) \left\{ E[V|H^{t_{\xi}}, S_2] - \text{bid}^{t_{\xi}}_{\xi, \mu} \right\} + \frac{1 - \theta - \mu}{3} \left\{ E[V|H^{t_{\xi}}] - \text{bid}^{t_{\xi}}_{\xi, \mu} \right\} + \theta \Pr(S_1|H^{t_{\xi}}) \left\{ E[V|H^{t_{\xi}}, S_1] - \text{bid}^{t_{\xi}}_{\xi, \mu} \right\}$$

Since $E\pi^{t_{\xi}}_b(\mu, \xi) = 0$, we know that

$$\mu \Pr(S_2|H^{t_{\xi}}) \left\{ E[V|H^{t_{\xi}}, S_2] - \text{bid}^{t_{\xi}}_{\xi, \mu} \right\} + \theta \Pr(S_1|H^{t_{\xi}}) \left\{ E[V|H^{t_{\xi}}, S_1] - \text{bid}^{t_{\xi}}_{\xi, \mu} \right\} \leq 0,$$

then $\mu \Pr(S_2|H^{t_{\xi}}) \left\{ E[V|H^{t_{\xi}}, S_2] - \text{bid}^{t_{\xi}}_{\xi, \mu} \right\} + \mu \Pr(S_1|H^{t_{\xi}}) \left\{ E[V|H^{t_{\xi}}, S_1] - \text{bid}^{t_{\xi}}_{\xi, \mu} \right\} < 0$ as $\mu < \theta$, which implies that

$$E[V|H^{t_{\xi}}] - \Pr(S_3|H^{t_{\xi}})E[V|H^{t_{\xi}}, S_3] < [1 - \Pr(S_3|H^{t_{\xi}})] \text{bid}^{t_{\xi}}_{\xi, \mu}$$

$$\leq E[V|H^{t_{\xi}}] - \Pr(S_3|H^{t_{\xi}})E[V|H^{t_{\xi}}]$$

This contradicts with $E[V|H^{t_{\xi}}] > E[V|H^{t_{\xi}}, S_3]$. Thus $\text{bid}^{t_{\xi}}_{\xi, \mu} > E[V|H^{t_{\xi}}]$. But this contradicts with $E[V|H^{t_{\xi}}] > E[V|H^{t_{\xi}}, S_3] \geq \text{bid}^{t_{\xi}}_{\xi, \mu}$, then $E[V|H^{t_{\xi}}, S_3] < \text{bid}^{t_{\xi}}_{\xi, \mu}$. And
\[ E[V|H^{\xi_1}, S_3] \leq \text{ask}_{\xi_1}^{t_\xi_1}, \text{ we know that an informed trader sells at period } t_\xi_1 \text{ if he receives signal } S_3. \]

First notice that in this case, \( \text{bid}_{\xi_1}^{t_\xi_1} > E[V|H^{\xi_1}] \).

Suppose not, i.e., \( \text{bid}_{\xi_1}^{t_\xi_1} \leq E[V|H^{\xi_1}] \). Then if an informed trader also sells when he receives signal \( S_2 \), as \( E^9_0 \mu, \xi \) = 0, if \( E[V|H^{\xi_1}, S_2] < \text{bid}_{\xi_1}^{t_\xi_1} \), we have

\[
\begin{align*}
&\mu \text{Pr}(S_2|H^{\xi_1}) \left\{ E[V|H^{\xi_1}, S_2] - \text{bid}_{\xi_1}^{t_\xi_1} \right\} + \mu \text{Pr}(S_3|H^{\xi_1}) \left\{ E[V|H^{\xi_1}, S_3] - \text{bid}_{\xi_1}^{t_\xi_1} \right\} \\
&+ \theta \text{Pr}(S_1|H^{\xi_1}) \left\{ E[V|H^{\xi_1}, S_1] - \text{bid}_{\xi_1}^{t_\xi_1} \right\} \\
&\leq 0.
\end{align*}
\]

Since \( \mu < \theta \) and \( E[V|H^{\xi_1}, S_1] > \text{bid}_{\xi_1}^{t_\xi_1} \), we have

\[
\begin{align*}
&\mu \text{Pr}(S_2|H^{\xi_1}) \left\{ E[V|H^{\xi_1}, S_2] - \text{bid}_{\xi_1}^{t_\xi_1} \right\} + \mu \text{Pr}(S_3|H^{\xi_1}) \left\{ E[V|H^{\xi_1}, S_3] - \text{bid}_{\xi_1}^{t_\xi_1} \right\} \\
&+ \mu \text{Pr}(S_1|H^{\xi_1}) \left\{ E[V|H^{\xi_1}, S_1] - \text{bid}_{\xi_1}^{t_\xi_1} \right\} < 0,
\end{align*}
\]

i.e., \( E[V|H^{\xi_1}] < \text{bid}_{\xi_1}^{t_\xi_1} \), contradicting with \( \text{bid}_{\xi_1}^{t_\xi_1} \leq E[V|H^{\xi_1}] \).

If \( E[V|H^{\xi_1}, S_2] \geq \text{bid}_{\xi_1}^{t_\xi_1} \), we have

\[
\begin{align*}
&\mu \text{Pr}(S_2|H^{\xi_1}) \left\{ E[V|H^{\xi_1}, S_2] - \text{bid}_{\xi_1}^{t_\xi_1} \right\} + \theta \text{Pr}(S_1|H^{\xi_1}) \left\{ E[V|H^{\xi_1}, S_1] - \text{bid}_{\xi_1}^{t_\xi_1} \right\} \leq 0.
\end{align*}
\]

Since \( \mu < \theta \) and \( E[V|H^{\xi_1}, S_1] > \text{bid}_{\xi_1}^{t_\xi_1} \), we have

\[
\begin{align*}
&\mu \text{Pr}(S_3|H^{\xi_1}) \left\{ E[V|H^{\xi_1}, S_3] - \text{bid}_{\xi_1}^{t_\xi_1} \right\} + \mu \text{Pr}(S_1|H^{\xi_1}) \left\{ E[V|H^{\xi_1}, S_1] - \text{bid}_{\xi_1}^{t_\xi_1} \right\} < 0,
\end{align*}
\]

then

\[
\begin{align*}
E[V|H^{\xi_1}] - \text{Pr}(S_2|H^{\xi_1}) E[V|H^{\xi_1}, S_2] &< [1 - \text{Pr}(S_2|H^{\xi_1})] \text{bid}_{\xi_1}^{t_\xi_1} \\
&\leq [1 - \text{Pr}(S_2|H^{\xi_1})] E[V|H^{\xi_1}],
\end{align*}
\]

i.e., contradicting with \( E[V|H^{\xi_1}, S_2] \leq E[V|H^{\xi_1}] \).

Thus \( \text{bid}_{\xi_1}^{t_\xi_1} > E[V|H^{\xi_1}] \).

Notice that when \( \text{ask}_{\xi_1}^{t_\xi_1} > E[V|H^{\xi_1}] \), an informed trader buys only if he receives
signal \( S_1 \), then

\[
as_{\xi, \mu} = \frac{\mu Pr(S_1|H^{t+1})E[V|H^{t+1}, S_1] + (\xi - \mu)Pr(S_3|H^{t+1})E[V|H^{t+1}, S_3] + \frac{1}{3}\xi E[V|H^{t+1}]}{\mu Pr(S_1|H^{t+1}) + (\xi - \mu)Pr(S_3|H^{t+1}) + \frac{1}{3}\xi},
\]

which strictly increases with \( \mu \). Thus \( as_{\xi, \mu} > E[V|H^{t+1}] \) if and only if

\[
\frac{\mu Pr(S_1|H^{t+1})E[V|H^{t+1}, S_1] + (\xi - \mu)Pr(S_3|H^{t+1})E[V|H^{t+1}, S_3] + \frac{1}{3}\xi E[V|H^{t+1}]}{\mu Pr(S_1|H^{t+1}) + (\xi - \mu)Pr(S_3|H^{t+1}) + \frac{1}{3}\xi} > E[V|H^{t+1}].
\]

As for any \( \mu < \frac{1}{2}\xi \), we know

\[
\lim_{\mu \to \frac{1}{2}\xi} as_{\xi, \mu} = \frac{\frac{1}{2}\xi Pr(S_1|H^{t+1})E[V|H^{t+1}, S_1] + \frac{1}{2}\xi Pr(S_3|H^{t+1})E[V|H^{t+1}, S_3] + \frac{1}{3}\xi E[V|H^{t+1}]}{\frac{1}{2}\xi Pr(S_1|H^{t+1}) + \frac{1}{2}\xi Pr(S_3|H^{t+1}) + \frac{1}{3}\xi} = \frac{\frac{1}{2}\xi \{ E[V|H^{t+1}] - Pr(S_2|H^{t+1})E[V|H^{t+1}, S_2]\} + \frac{1}{3}\xi E[V|H^{t+1}]}{\frac{1}{2}\xi [1 - Pr(S_2|H^{t+1})] + \frac{1}{3}\xi},
\]

then for any given \( \xi \in (0, 1) \), there exists \( \mu_\xi \in (0, \frac{1}{2}\xi) \) such that if and only if \( \mu \in \{ \mu_\xi, \frac{1}{2}\xi \} \), we have \( as_{\xi, \mu} > E[V|H^{t+1}] \), which implies that a bubble occurs. Otherwise, when \( \mu \in (0, \mu_\xi) \), \( as_{\xi, \mu} \leq E[V|H^{t+1}] < bid_{\xi, \mu} \), that is, a crossed market occurs. \( \blacksquare \)

4 Conclusions

This paper provides one explanation for the existence of crossed markets for informationally efficient prices under perfect competition in financial markets. The presence of naive traders make it possible for the market maker to make positive profit from trading with noise traders, which requires bid prices to be greater than the market values and the ask prices to be lower than the market values under the 0-profit condition imposed by perfect competition. Also, this paper points out that if we see crossed market occur for informationally efficient prices in competitive markets, we can conclude that there are traders with multiple types of information disadvantages compared to the market maker.
Appendix

Lemma 1 For any \( S \in S, \mu, \theta \in (0, 1), \) and history \( H^t, E[V|S, H^t] - E[V|H^t] \) has the same sign as \( E \)
\[
q_1^i q_2^j [Pr(S|V_2) - Pr(S|V_1)] + q_2^j [Pr(S|V_3) - Pr(S|V_2)] + 2q_3^i [Pr(S|V_3) - Pr(S|V_1)].
\]

Proof. (Lemma 1)
\[
E[V|H^t, S] - E[V|H^t] = \left[ VPr(V_2|H^t, S) + 2VPr(V_3|H^t, S) \right] - \left[ VPr(V_2|H^t) + 2VPr(V_3|H^t) \right]
= V \left[ Pr(V_2|H^t, S) - q_2^j \right] + 2V \left[ Pr(V_3|H^t, S) - q_3^i \right].
\]
Since for any \( i = 1, 2, 3, \)
\[
Pr(V_i|H^t, S) = \frac{Pr(V_i)}{Pr(H^t, S)} = \frac{Pr(V_i)}{Pr(H^t)Pr(V_i)} = \frac{Pr(V_i)}{\sum_{j=1}^3 Pr(V_j)Pr(V_i)} = \frac{Pr(V_i)}{\sum_{j=1}^3 Pr(V_j)}q_j^i,
\]
we have \( E[V|H^t, S] - E[V|H^t] = V \left[ \frac{Pr(V_2)\sum_{j=1}^3 q_j^i}{\sum_{j=1}^3 Pr(V_j)q_j^i} - q_2^j \right] + 2V \left[ \frac{Pr(V_3)\sum_{j=1}^3 q_j^i}{\sum_{j=1}^3 Pr(V_j)q_j^i} - q_3^i \right]. \)
and simple computation leads to the conclusion. ■

Corollary 1 If a signal \( S \in \{S_1, S_2, S_3\} \) is neutral, then for any history, \( E[V|H^t, S] = E[V|H^t] \).

Lemma 2 For any signal \( S \in S \) and \( \mu, \theta \in (0, 1), E[V|S] \) is less than \( E[V] \) if and only if \( S \) has a negative bias, and \( E[V|S] \) is greater than \( E[V] \) if and only if \( S \) has a positive bias. Thus, \( S_1 \) has negative bias and \( S_3 \) has positive bias.

Proof. (Lemma 2) For any given signal \( S \in \{S_1, S_2, S_3\}, \) we have
\[
E[V|S] = \sum_{i=1}^3 V_i Pr(V_i|S) = VPr(V_2|S) + 2VPr(V_3|S) = V \frac{Pr(V_2, S)}{Pr(S)} + 2V \frac{Pr(V_3, S)}{Pr(S)}
\]
and \( E[V] = \sum_{i=1}^3 V_i Pr(V_i) = VPr(V_2) + 2VPr(V_3), \) then \( E[V|H^t] > E[V] \) is equivalent to
\[
\frac{Pr(V_2, S)}{Pr(S)} + 2\frac{Pr(V_3, S)}{Pr(S)} > Pr(V_2) + 2Pr(V_3) = Pr(V_2) + Pr(V_3) + Pr(V_1) = 1.
\]
\(^8\)If \( \theta = 0, \) this lemma is the same as Lemma 1 in Park and Sabourian (2011).
\(^9\)If \( \theta = 0, \) this lemma is the same as Lemma 2 in Park and Sabourian (2011).
That is, \( Pr(V_2, S) + 2Pr(V_3, S) > Pr(S) = Pr(V_1, S) + Pr(V_2, S) + Pr(V_3, S) \). Then \( Pr(V_3, S) > Pr(V_1, S) \), therefore \( Pr(V_3|S) > Pr(V_1|S) \). Thus \( S \) has a positive bias.

Similar reasoning applies to the case that \( E[V|S] < E[V] \) if and only if \( S \) has a negative bias. ■

**Lemma 3** For any history \( H^t \) and \( \mu, \theta \in (0, 1) \), \( E[V|H^t] > E[V] \) if and only if \( q_{\mu}^t > q_{\theta}^t \). \( E[V] > E[V|H^t] \) if and only if \( q_{\mu}^t > q_{\theta}^t \).

**Proof. (Lemma 3)** As \( E[V|H^t] = \sum_{i=1}^{3} V_i Pr(V_i|H^t) = VPr(V_2|H^t) + 2VPr(V_3|H^t) \), we have \( E[V|H^t] > E[V] \) is equivalent to \( Pr(V_2|H^t) + 2Pr(V_3|H^t) > Pr(V_2) + 2Pr(V_3) = 1 \).

Simple calculation shows that this implies that \( Pr(V_2, H^t) + 2Pr(V_3, H^t) > Pr(H^t) = Pr(V_1, H^t) + Pr(V_2, H^t) + Pr(V_3, H^t) \), then \( Pr(V_3, H^t) = \frac{Pr(V_2|H^t)}{Pr(H^t)} > Pr(V_1, H^t) = \frac{Pr(V_3|H^t)}{Pr(H^t)} \).

Thus \( Pr(V_3|H^t) > Pr(V_1|H^t) \), that is, \( q_{\mu}^t > q_{\theta}^t \).

The case that \( E[V|H^t] < E[V] \) if and only if \( q_{\mu}^t < q_{\theta}^t \) can be deduced in the same way. ■

**Proposition 3** For any public history \( H^t \), \( \max \{E[V|H^t, S_1], E[V|H^t, S_2], E[V|H^t, S_3]\} > \max \{\text{bid}^d, \text{ask}^d\} \) and \( \min \{E[V|H^t, S_1], E[V|H^t, S_2], E[V|H^t, S_3]\} < \min \{\text{bid}^d, \text{ask}^d\} \).

**Proof. (Proposition 3)** We ignore the trivial case in which \( E[V|H^t, S_1] = E[V|H^t, S_2] = E[V|H^t, S_3] = E[V|H^t] \). In this case, \( \text{ask}^d = E[V|H^t] = \text{bid}^d \). We focus on the case at least one of the three expectations \( E[V|H^t, S_1], E[V|H^t, S_2], E[V|H^t, S_3] \) is strictly greater than \( E[V|H^t] \), and one of them is strictly less than \( E[V|H^t] \). In other words, we focus on the case in which \( \max \{E[V|H^t, S_1], E[V|H^t, S_2], E[V|H^t, S_3]\} > E[V|H^t] \) and \( \min \{E[V|H^t, S_1], E[V|H^t, S_2], E[V|H^t, S_3]\} < E[V|H^t] \).

Suppose that \( \text{bid}^d \geq E[V|H^t, S], \forall S \in \{S_1, S_2, S_3\} \). Then \( \text{bid}^d \geq E[V|H^t, S_1] \) and \( \text{bid}^d > E[V|H^t] \), and an informed trader sells to the market maker regardless what signal he receives.

When purchasing one unit of the asset at \( \text{bid}^d \) from a trader, the market maker’s expected profit is \( Pr(S_1|H^t) (E[V|H^t, S_1] - \text{bid}^d) \leq 0 \) if the trader is a naive trader. Since she also makes a strict negative profit if he trades with an informed traders, the market market maker must have a strict positive gain from trading with a noise trader at

\[10\text{If} \theta = 0, \text{this lemma is the same as Lemma 3 in Park and Sabourian (2011).} \]
As her expected profit from purchasing one unit of the asset from a noise trader is $1 - \theta - \mu$, we have $E[V|H^t] > bid^t$, contradicting with $bid^t > E[V|H^t]$. Thus there exists at least one signal $S \in \{S_1, S_2, S_3\}$ such that $bid^t < E[V|H^t, S]$, then $\max\{E[V|H^t, S_1], E[V|H^t, S_2], E[V|H^t, S_3]\} > bid^t$.

Similarly, if $ask^t \geq \max\{E[V|H^t, S_1], E[V|H^t, S_2], E[V|H^t, S_3]\}$, then $ask^t > E[V|H^t, S]$, which implies that only a naive trader who receives signal $S_3$ or a noise trader will buy from the market maker. Since the market maker makes positive profit $ask^t - E[V|H^t, S_3]$ from selling one unit of the asset to a naive trader, then he must make negative profit from selling one unit of the asset to a noise trader, which implies $ask^t \leq E[V|H^t, S]$, contradicting with $ask^t > E[V|H^t, S]$. So there exists at least one signal $S \in \{S_1, S_2, S_3\}$ such that $ask^t < E[V|H^t, S]$, then $\max\{E[V|H^t, S_1], E[V|H^t, S_2], E[V|H^t, S_3]\} > ask^t$.

Now we have $\max\{E[V|H^t, S_1], E[V|H^t, S_2], E[V|H^t, S_3]\} > \max\{bid^t, ask^t\}$, Similarly we also have $\min\{E[V|H^t, S_1], E[V|H^t, S_2], E[V|H^t, S_3]\} < \min\{bid^t, ask^t\}$. ■

Reference


