Voting with Strategic Heterogeneity

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2013

Abstract
Consider a model with asymmetric information of the type analyzed in Feddersen-Pesendorfer (1996,1998). There are two alternatives and two states of the world; voters receive private signals about the true state of the world; conditional on the state of the world, all voters have the same preferences over alternatives. The twist introduced by this paper is that with probability $1-\theta$ a voter is non-strategic, meaning that this voter votes so as to maximize his utility conditioning on his private signal, but not conditioning on information that may be contained in the votes of other people. The presence of non-strategic voters improves information aggregation in the sense that the information of the non-strategic voter is directly reflected in their votes. But the presence of non-strategic voters exacerbates the “swing voter’s curse”. The information revealed by non-strategic voters push strategic voters to put less weight on their own signals. If the prior probability that a voter is non-strategic is high enough, and the total population is large enough, strategic voters ignore their own private signals entirely.\footnote{Youzong Xu acknowledges financial support from the Center for Research in Economics and Strategy (CRES), in the Olin Business School, Washington University in St. Louis.}

1 Introduction
Consider a 2(alternatives) × 2(states) voting model with asymmetric information, as in the framework analyzed by Feddersen-Pesendorfer (1998,1996). Voters cannot observe the true state of the world, instead, each voter receives a private and noisy signal about the true state. Compared to Feddersen-Pesendorfer (1998,1996) and other related literature, the key twist introduced in this paper is the presence of non-strategic voters. Non-strategic voters are those who only
look at their own private signals when updating information about the true state. *Strategic* voters, on the other hand, use the information contained by other voters’ actions, together with their own signals, to update their information about the true state, conditioning on being pivotal\(^2\).

Specifically, consider a two-state voting model in which the issue is to approve a policy using a \(q\)-rule. In state \(W\), the policy works, and in state \(NW\), the policy does not work. Voters cannot observe the true state but only noisy signals \(w\) and \(nw\).\(^3\) Each voter must vote (no abstention) and can vote to approve \((A)\) or reject \((R)\). Voters prefer the policy to be approved if the true state is \(W\) and to be rejected if the true state is \(NW\). Given the total population of voters, \(N\), the policy is approved if and only if more than \(qN\) voters vote \(A\), in other words, we need at least \((2q - 1)N\) more votes for \(A\) than for \(R\) to have the policy approved.\(^4\)

There are two types of voters in the population. A non-strategic voter always follows his signal, in the sense that if he receives signal \(w\), he votes to approve \((A)\), and if he receives signal \(nw\), he votes to reject the policy \((R)\). In other words, non-strategic voters always vote informatively.

The existence of strategic heterogeneity, with both strategic and non-strategic voters in the same population, is supported by empirical work. See, for example, Guarnaschelli, McKelvey, and Palfrey (2000), who show that a strictly positive proportion of subjects in their experiments act less strategically than others. Say that voter population is *strategically heterogeneous* if some voters are strategic and some are non-strategic.

Compared to the *fully-strategic* model, in which all voters are strategic, the presence of non-strategic voters exacerbates the “swing voter’s curse”, in the sense that when there is strategic heterogeneity in the population, strategic voters put less weight on their own signals. And if the prior probability that a voter is non-strategic is high enough, and the total population is large enough, strategic voters ignore their own private signals entirely. This does not happen if all voters are strategic, or some of them are partisans. In these two latter models, strategic voters always put strictly positive weight on their own signals, as pivotalities in these models can never be rich enough in information revelation.

Because of this, if there is strategic heterogeneity among voters, when the total population is large, strategic voters always have higher tendency to vote \(A\), compared to the full-strategic model. But the influence of strategic heterogeneity on collective decision in the overall

\(^{2}\)A voter is pivotal if, given other voters’ actions, the collective decision changes if this pivotal voter changes his action.

\(^{3}\)Assume that \(\Pr(w|W) = \Pr(nw|NW) = p \in (\frac{1}{2}, 1)\).

\(^{4}\)\(qN\) and \((2q - 1)N\) are assumed to be integers.
information aggregation level is complicated.

To see how strategic heterogeneity affects overall information aggregation, let’s define some notation. Consider a model with strategic heterogeneity under a $q$-rule. Let $\theta$ denote the prior probability that a voter is strategic. Let $S \in \{W, NW\}$ be the true state. Let $\gamma^q_s$ denote the probability that a strategic voter votes $A$. Let $\beta^q_s$ denote the prior probability that a voter, who may be strategic or non-strategic, votes $A$. Similar definitions apply to $\gamma^q_F$ and $\beta^q_F$ in the fully-strategic model. The superscript $q$ indicates the given value of $q$ and the superscript $F$ always indicates terms in the the full-strategic case. This notation is used throughout the paper. Then

$$\beta^q_s = (1 - \theta) \cdot Pr(w|S) + \theta \cdot \gamma^q_s.$$ 

Since non-strategic voters always vote informatively, $Pr(w|S)$ is the probability that a non-strategic voter votes $A$ in state $S$. Then first term of $\beta^q_s$, $(1 - \theta) \cdot Pr(w|S)$, represents how a non-strategic voter’s private information (his signal) enters the overall information aggregation. Call this the direct information effect. As $Pr(w|S)$ is exogenous, a drop in $\theta$ causes $(1 - \theta) \cdot Pr(w|S)$ to increase. That is, a lower $\theta$ implies a stronger direct information effect.

The second term, $\theta \cdot \gamma^q_s$, is the probability that a voter is strategic and votes $A$. The dependence of this term on $\theta$ is ambiguous. To see this, recall that at pivotality, there are $(2q - 1)N$ more votes for $A$ than for $R$. When the voter population is strategically heterogeneous, a strategic voter knows that some of these $(2q - 1)N$ votes for $A$ are made by non-strategic voters who receive signal $w$. Compared to pivotality in the fully-strategic model, this implies that more voters receive signal $w$, which means that there is more information favoring state $W$, since $Pr(w|W) > \frac{1}{2}$. Then a strategic voter has stronger motivation to vote $A$. The lower is $\theta$, the more information favoring state $W$ revealed by pivotality, regardless of what the true state is. Thus $\gamma^q_s$ is always higher than $\gamma^q_F$, $\forall S \in \{W, NW\}$. Strategic heterogeneity affects $\beta^q_s$ indirectly through $\gamma^q_s$ in this process. Call this the indirect information effect.

Since $\theta$ and $\gamma^q_s$ go in opposite directions, how their product changes with $\theta$ is not straightforward. Together with the explicit increase of $(1 - \theta) \cdot Pr(w|S)$ caused by the decrease of $\theta$ through the direct information effect, how the overall

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information aggregation is affected by the change of $\theta$ becomes complicated. For more detailed discussions please see Section 6 and Section 7.

Though most literature on this topic assume strategic homogeneity among voters, i.e., all voters are strategic, or all of them are non-strategic, there are some models that consider another type of strategic heterogeneity. In these models, some voters are strategic as defined in my model, but others are partisan who vote for certain alternatives regardless of their signals. The fundamental difference between partisans in this literature and the non-strategic voters in my paper is with information aggregation. A partisan’s action reveals nothing about his private information, while a non-strategic voter’s action reveals all his private information. Such characteristics of partisans affect the swing voter’s curse in the opposite direction as how non-strategic voters do, in the sense that strategic voters always place strictly positive weight on their own signals.

For most of the paper, voters cannot abstain. I consider voluntary voting with abstention briefly in Section 9.

This paper is organized as follows. Section 2 reviews some important results in the related literature. Section 3 is about the basic setup of the model with strategic heterogeneity. Section 4 introduces all the related results in a traditional full-strategic model as benchmarks. Section 5 focuses on how strategic heterogeneity affect strategic voters’ behaviors, where Section 6 discusses the influence of strategic heterogeneity imposed on information aggregation. Section 7 suggests some hints in looking for good voting rules for collective decisions, based on the results in Section 6. Section 9 discusses voluntary voting in which abstention is allowed. Section 10 concludes, and also shows some potential extensions. All proofs are in the appendix.

2 Related Literature

The proof of Condorcet Jury Theorem and the extensive literature (e.g., Berg 1993; Ladha 1992; and Young 1988) on this theorem assumes that every individual votes nonstrategically or sincerely, and behaves the same whether he acts as a dictator or participates in a group making decision. This assumption was challenged since 1990s, by both of theoretical research and empirical data.

Theoretic research, for example, Austen-Smith and Banks (1996) and Feddersen and Pesendorfer (1997), point out that nonstrategic voting is not consistent with Nash equilibrium behavior if all voters are voting strategically. Feddersen and Pesendorfer (1996) have similar results in a model with both strategic voters and partisan.

Since 1990s, most of the literature on voting theory takes strategic voting as
axiomatic, most of which assume that all voters act strategically. For example, Feddersen and Pesendorfer (1998), Coughlan (2000), Kojima and Takagi (2010), and etc. Even in some literature that allows some more general assumptions into the traditional strategic voting framework, including communications among voters (Gerardi and Yariv (2007)), or combination of alternatives (Ahn and Oliveros (2012)), voters are assumed to be strategic.

But it is pretty natural for us to think that not all voters are as strategic as described in the literature. We may think that in reality, some individuals are strategic, while some others are less sophisticated. There is empirical research supporting such heterogeneity in strategic level. Guarnaschelli, McKelvey, and Palfrey (2000) finds evidence of strategic voting behavior in experiment data. Their data shows that a large proportion, between 50% and 70%, of subjects vote less strategically as others.

In some literature, such “less-strategic voters” are classified as partisans who always choose certain alternative and ignore their signals entirely. But evidence found in experiments shows that some voters may be “in between” being strategic and partisan. See for example, Ladha, Miller and Oppenheimer (1996) show that some voters tend to always follow their signals so their actions disclose their information truthfully. Similar truthful actions are also found in jury trials, in which some jurors concern only the evidence presented in court\(^6\). Such a specific heterogeneity, by which I mean a mixed population of strategic voters and those “truth-telling” ones, has rarely been analyzed. That is the motivation for this paper.

3 Basic Setup

In a referendum, \(N \geq 2\) voters are voting to decide whether to approve or reject a policy. As voting is mandatory, each voter has to vote and can vote to approve (\(A\)) or reject (\(R\)) this policy. The collective decision of this referendum is determined by a \(q\)-rule, that is, the policy will be approved (\(AP\)) if and only if at least \(qN\) voters vote \(A\), otherwise it will be rejected (\(RE\)).

There are two states with the same probability: state \(W\), in which adopting the policy is optimal (\(W\) for the policy “work”), and state \(NW\), in which rejecting the policy is optimal (\(NW\) for “not work”). Voters can not observe the true state but only noisy signals \(w\) and \(nw\). Assume that \(Pr(w|W) = Pr(nw|NW) = p \in (\frac{1}{2}, 1)\),\(^7\) where \(Pr(w|W)\) and \(Pr(nw|NW)\) denote the probabilities that a

\(^6\)They are called “conscious” jurors in some literature.

\(^7\)The main thrust of our results will not change if these two probabilities do not equal to each other.
voter receives the correct signal corresponding to the true state, respectively. Given any state $S \in \{W, NW\}$, the distributions of signals $s \in \{w, nw\}$ received by voters are i.i.d.

Say a voter is **pivotal** that the policy will be approved if and only if he votes $A$. Let $piv$ denote the event that a strategic voter is pivotal.

Say a voter is voting **informatively** if he votes $A$ when she receives signal $w$, and votes $R$ when he receives $nw$. Such a strategy is equivalent to the “truth-telling” strategy in a game in which the collective decision is made based on the signals that players report.

Voters are strategically heterogeneous. Some voters are **non-strategic**, always vote informatively. Other are **strategic**, they vote for the alternative that maximizes their expected utility, which turns out to be equivalent to maximizing expected utility conditioned on being pivotal. The type of a voter is private information known by herself only. But the probability that a voter is strategic, denoted by $\theta \in (0, 1)$, is public information.

This assumption on strategic heterogeneity catches two characteristics of voters in reality: 1) not all voters have the same level of rationality or adopt the same method in making their decisions, which has been proved by empirical researches; and 2) voters may not know each other’s methods in decision-making, especially in voting systems with large populations of voters.

When the result $\Omega$ of the referendum is decided and the true state is $S$, a strategic voter’s utility is $U(\Omega, S)$ with $U(AP, W) = U(RE, NW) = 0$, $U(AP, NW) = -\alpha$, and $U(RE, W) = -(1 - \alpha)$. Assume $\alpha \in (\frac{1}{2}, 1)$, which implies that approving the policy when it does not work is the worst mistake.

There is realistic and intuitive support for adopting this asymmetry of disutilities caused by mistakes. First, it is natural to think that different mistakes cause different levels of disutility. For instance, consider the Greek referendum in which Greeks vote to decide whether Greece should withdraw from the European Union. Once withdraw from EU, it is very difficult for Greece to rejoin it. But if Greece decides to stay in EU, Greece can still withdraw any time in the future. If time discount factor is strictly less than 1, staying in EU when Greece should withdraw leads to less disutility than withdrawing from EU by mistake.

Jury trial is another good example supporting this asymmetry. Let $W$ correspond to the state in which the defendant is guilty, and $NW$ correspond to the state in which the defendant is innocent. And let the approval of this policy implies convicting the defendant, which refusal of the policy implies acquitting him. The common idea in the jury system, that “Trial by jury...is a process to make it as sure as possible that no innocent man is convicted”\(^8\), implies that people prefer acquitting a guilty defendant to convicting an innocent one, which

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\(^8\)Lord Devlin. See Klaven and Zeisel, 1996, 190.
implies asymmetry in disutilities caused by wrong verdicts.

Another reason for us to use this assumption is that it eliminates the probability that strategic voters always vote to approve, or reject, the policy regardless of their signals even when the population of voters is small, that is, when a single strategic voter has comparatively larger manipulation power. So this assumption on asymmetry of utilities captures one of the main motivations that strategic voters act strategically: to avoid or minimize the disutilities caused by wrong collective decisions.

Under this framework, a strategic voter prefers approving the policy to rejecting it if and only if the probability that the true state is $W$ is at least $\alpha$. Thus $1 - \alpha$ can be considered as a strategic voter’s threshold level of “reasonable doubt” as we can see that a voter will vote to approve the policy if and only if his doubt that the true state is $NW$ is lower than his reasonable doubt level.

Before proceeding, a couple of points need to be clarified and emphasized. First, all voters vote simultaneously and there is no communication among all voters, so collusion is impossible. Also, though $N$, $q$, $\theta$, and $p$ are common information, voters do not know each other’s type or signals.

Second, since I focus on symmetric Nash equilibria, there is no need to identify particular voter in the notation. Also, as non-strategic voters always follow the same strategy, informative voting, I can just adopt the strategies of strategic voters to represent strategies in any symmetric Nash equilibrium.

Let $\sigma(s)$ denote the probability that a strategic voter votes $A$ when he receives signal $s \in \{w, nw\}$. Then a symmetric strategy of strategic voters can be represented by $(\sigma(w), \sigma(nw))$, which is the strategy of one voter.

Let $\gamma_w = p\sigma(w) + (1 - p)\sigma(nw)$ and $\gamma_{NW} = (1 - p)\sigma(w) + p\sigma(nw)$ denote the probabilities that a strategic voter votes to approve the policy when the true state is $W$ or $NW$, respectively. Here the subscripts represent the true state. Say a strategic voter’s strategy $(\sigma(w), \sigma(nw))$ is responsive if $\gamma_w \neq \gamma_{NW}$.

There are five different strategies that can form symmetric Nash equilibria. The first one, $(\sigma(w), \sigma(nw)) = (1, 1)$, I call $H$. It is the one with the highest $\gamma_w$ and $\gamma_{NW}$. Strategic voters using this strategy vote $A$ regardless of their own signals.

The symmetric strategy with the lowest $\gamma_w$ and $\gamma_{NW}$ is $(\sigma(w), \sigma(nw)) = (0, 0)$, I call this $O$. Strategic voters who adopt this strategy vote $R$ regardless what signals they receive.

Notice that neither $O$ nor $H$ is responsive, as in both cases, $\gamma_w = \gamma_{NW}$. There are three responsive strategies that may constitute symmetric Nash equilibria.

Strategy $(\sigma(w), \sigma(nw)) = (1, 0)$, called $M$, is the one in which strategic voters vote informatively.

Two other types of symmetric strategies are $I$, in which $\sigma(w) = 1$ and $\sigma(nw) \in (0, 1)$, and $II$, in which $\sigma(w) \in (0, 1)$ and $\sigma(nw) = 0$. 7
Nash equilibrium formed by strategy $i$ where $i \in \{O, I, II, M, H\}$ is called a Type $i$ Nash equilibrium.

One might wonder why there could not be other strategies that form symmetric Nash equilibria, for instance, the strategy that $\sigma(w) = 0$ and $\sigma(nw) > 0$.

Theorem 1 shows that the answer to this question is no. This theorem shows that strategy $(\sigma(w), \sigma(nw))$, where $\sigma(w) \in (0, 1)$ and $\sigma(nw) \in (0, 1]$, cannot be a symmetric Nash equilibrium strategy.

**Theorem 1** Consider any given $\{q, p, \theta, \alpha\}$ with $q, p, \alpha \in \left(\frac{1}{2}, 1\right)$ and $\theta \in (0, 1)$. Suppose a symmetric strategy $(\sigma(w), \sigma(nw))$ is a Nash equilibrium strategy, then if $\sigma(nw) > 0$, we have $\sigma(w) = 1$; and if $\sigma(w) < 1$, we have $\sigma(nw) = 0$.

Let $Pr_N(\text{AP}|\text{NW})$ denote the probability that the policy is proved when it is not going to work, and $Pr_N(\text{RE}|W)$ denote the probability that the policy is rejected when it will work. Here subscript $N$ denote the population.

**Definition 1** Say information is aggregated effectively if in equilibrium, the probabilities that mistakes happen diminish to zero when the population size of voters increases to infinity.\(^9\) Numerically, effective information aggregation means:

$$\lim_{N \to \infty} Pr_N(\text{AP}|\text{NW}) = \lim_{N \to \infty} Pr_N(\text{RE}|W) = 0.$$  

Say a $q$, or a $q$-rule, is informationally effective, if under this $q$-rule, information is aggregated effectively.

**4 Benchmark Model: Fully-Strategic Model**

To see how strategic heterogeneity affects strategic voters behavior and information aggregation, we need a benchmark, so let’s look at the fully-strategic model first, by which we mean every voter is strategic, that is, $\theta = 1$.

**Lemma 1** For any given $\{q, p, \theta, \alpha\}$ with $q, p, \alpha \in \left(\frac{1}{2}, 1\right)$ and $\theta \in (0, 1)$, there is a unique, finite $N^* \geq 2$ s.t.

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\(^9\)In some literature, an equivalent terminology of effective aggregation of information is information equivalence.
\[ \frac{1}{1 + \left( \frac{1-p}{p} \right)^{(2q-1)(N+1)-2}} > \alpha \quad \text{and} \quad \frac{1}{1 + \left( \frac{1-p}{p} \right)^{(2q-1)N^*+1}} \leq \alpha. \]

**Proposition 1** Consider any given \( \{N, q, p, \theta, \alpha\} \) with \( q, p, \alpha \in \left( \frac{1}{2}, 1 \right) \) and \( \theta \in (0, 1) \), and \( N > \max\{N^*, \frac{1}{1-q}\} \), then in the fully-strategic model, the only symmetric Nash equilibria are of

(1.1) Type O or H;

(1.2) Type I, in which strategic voters adopt the strategy profile

\[ (\sigma^*_q(w), \sigma^*_q(nw)) = \left( \frac{1 - p B^*_q}{p - (1 - p) B^*_q} \right) \]

where \( B^*_q = \left[ \left( \frac{1-\alpha}{\alpha} \right) \cdot \left( \frac{1-p}{p} \right)^{N^*+1} \right]^{\frac{1}{N^*+1}} = \left( \frac{1-\alpha}{\alpha} \right)^{\frac{1}{N^*+1}} \cdot \left( \frac{1-p}{p} \right)^{\frac{N^*}{N^*+1}}. \] (Feddersen and Pesendorfer, 1998, Appendix B.)

The existence of Nash equilibrium of Type I comes directly from Proposition 1. The intuition for the existence of a Type H (O) equilibrium is straightforward in the sense that if all other voters vote \( A (R) \), then a single voter is never pivotal under \( q \)-rule with \( N > \frac{1}{1-q} \), so he is indifferent between voting \( A \) and voting \( R \).

**Proposition 2** In the Fully-Strategic Model, for any given \( \{q, p, \alpha\} \) with \( q, p, \alpha \in \left( \frac{1}{2}, 1 \right) \),

(2.1) information is aggregated effectively in the Type I Nash equilibrium; in other words, any \( q \)-rule is informationally effective in this Nash equilibrium; (Feddersen and Pesendorfer, 1998, Proposition 2.)

(2.2) information is not aggregated effectively in either Type O or H Nash equilibria; in other words, there is no informationally effective \( q \)-rule in these two Nash equilibria.

### 5 Strategic Voters’ Behaviors

Now we turn to the more general and realistic case in which voter population is strategically heterogeneous, that is, numerically, \( \theta \in (0, 1) \).

It is intuitive for us to think that strategic voters’ behavior is correlated with the level of \( q \) and \( \theta \), which is supported by further analysis. Actually, there exist
cutoff points of $q$ and $\theta$, such that strategic voters behave differently on different sides of these cutoff point values. First I look at the cutoff point of $\theta$ defined in Lemma 2, from which we can see that it depends on the values of $p$ and $q$ only, so call it $\theta_{qp}$.

Lemma 2 For any given $\{q,p\}$ with $q,p \in (\frac{1}{2}, 1)$, there exists a unique $\theta \in (0, 1)$, denoted as $\theta_{qp}$, such that

1. if $\theta < \theta_{qp}$, \(\frac{1-q}{p} > \frac{1-(1-\theta)p}{1-(1-\theta)(1-p)}\),

2. if $\theta > \theta_{qp}$, \(\frac{1-q}{p} < \frac{1-(1-\theta)p}{1-(1-\theta)(1-p)}\).

Then $\theta_{qp} = p \cdot \left(\frac{1-q}{p} - (1-p)\right) \frac{1}{(1-p) \frac{1-q}{p}}$.

Lemma 3 For any given $\{q,p\}$ with $q,p \in (\frac{1}{2}, 1)$, $\theta_{qp} > \frac{2-p}{1-p}$.

Let $\beta_{WN}^q = (1-\theta)p + \theta p \gamma_{WN}^q$ denote the ex ante probability that a voter votes to approve this policy when the true state is that the policy will work, i.e., $\beta_{WN}^q$ is the probability that a voter votes $A$ when the true state is $W$. Similarly, $\beta_{NW}^q = (1-\theta)(1-p) + \theta (1-p) \gamma_{NWN}^q$ is the ex ante probability that a voter votes $A$ when the true state is $NW$. As a reminder, $\gamma_{SN}$ is the ex ante probability that a strategic voter votes $A$ when the true state is $S \in \{NW,W\}$ and the population of voters is $N$. Here the superscription represents the value of $q$, and the subscription $WN$ and $NW$ represents the true state and the population of voters.

Before proceeding further ahead, recall that strategic voters update their expectation on the true state by using information revealed by pivotality, and what really matters in information revelation at pivotality are the numbers of votes on each alternative. Such numbers are determined by two factors: the total population of voters and the ex ante probability that a voter is strategic.

Lemma 4 For any given $\{q,p,\theta,\alpha\}$ with $q,p,\alpha \in (\frac{1}{2}, 1)$ and $\theta \in (0, 1)$, define $\hat{N}$ as following:

1. if $\theta < \theta_{qp}$, define $\hat{N}$ by $\sigma_{\hat{N}}^{q} \geq \theta > \sigma_{\hat{N}-1}^{q}(nw)$;

2. if $\theta > \theta_{qp}$, define $\hat{N} = \infty$. 

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Such $\hat{N}$ is unique.

Note that if $\theta > \theta_{qp}$, $\theta^{qF^*}(nw) < \theta$ for all $N$.

5.1 $\theta > \theta_{qp}$

As mentioned in early sections, a drop of $\theta$ from 1 to be strictly lower than 1 makes pivotality richer in information revelation. To see this more clear, recall that at pivotality, there are $(2q - 1)(N - 1)$ more votes for $A$ than for $R$, given total voter population $N$. A drop in $\theta$ implies a higher $1 - \theta$, which is the prior probability that a voter is non-strategic. An increase in $1 - \theta$ implies “larger” proportion of the $(2q - 1)(N - 1)$ votes are made by non-strategic voters who receive signal $w$, which means more information revealed favoring state $W$. Thus when $\theta$ drops from 1 to some level strictly below 1, more information support state $W$ is revealed by pivotality, and then strategic voters have higher tendency to vote $A$, regardless what signal they receive.

Notice that this relative richness in information revelation is related to the level of $\theta$, Theorem 2 and Theorem 3 clearly state how the level of $\theta$ affects strategic voters’ behaviors.

Theorem 2 Consider any given $\{q, p, \theta, \alpha\}$ with $q, p, \alpha \in (\frac{1}{2}, 1)$ and $\theta \in (0, 1)$. If $\theta > \theta_{qp}$, for any $N > \hat{N}^*$, there exists a unique symmetric Nash equilib-rium, in which the strategic voters adopt I strategy profile $(\sigma^q_N(w), \sigma^q_N(nw)) = (1, 1, \frac{1}{\theta} \cdot \sigma^q_{F^*}(nw))$, where $\sigma^q_{F^*}(nw) = \frac{pB^*_N - (1-p)}{p - (1-p)B^*_N}$ as defined in Proposition 1. That is, Type I Nash equilibrium is the unique symmetric Nash equilibrium when $N$ is large.

Since $\theta < 1$, we have $\sigma^q_N(nw) > \sigma^q_{F^*}(nw)$, together with $\sigma^q_N(w) = \sigma^q_{F^*}(w) = 1$, we know that in the Nash equilibrium of Type I, strategic voters have higher tendency to vote $A$ when some voters are non-strategic than when all voters are strategic. But as $\theta > \theta_{qp}$, by some simple calculation we can see that $\sigma^q_N(nw) < 1$, which implies that the increase in the richness of information revealed by pivotality caused by strategic heterogeneity is not great enough for strategic voters to ignore their own signals entirely.

5.2 $\theta < \theta_{qp}$

Lemma 5 For any given $\{q, p, \theta, \alpha\}$ with $q, p, \alpha \in (\frac{1}{2}, 1)$ and $\theta \in (0, 1)$, if $\theta < \theta_{qp}$, $N > \hat{N}^*$.
Theorem 3 Consider any given \( \{N, q, p, \theta, \alpha\} \) with \( q, p, \alpha \in (\frac{1}{2}, 1) \), \( \theta \in (0, 1) \), if \( \theta < \theta_{np} \), we have

(3.1) if \( \hat{N} > \hat{N}^* + 1 \), then for any \( N \) such that \( \hat{N}^* < N < \hat{N} \), the Type I Nash equilibrium, constructed by the I strategy \((\sigma_q^* (w), \sigma_q^* (nw)) = (1, \frac{1}{2} \cdot \sigma_{qF}^* (nw))\) is the unique symmetric Nash equilibrium. Here \( \sigma_{qF}^* (nw) \) is defined as in Theorem 2 and Proposition 1;

(3.2) for any \( N > \hat{N} \), Type H Nash equilibrium is the unique symmetric Nash equilibrium. In this Nash equilibrium, strategic voters adopt strategy \((\sigma_q^* (w), \sigma_q^* (nw)) = (1, 1)\).\(^{10}\)

When we compare the second part of this theorem to Theorem 2 we see that when the population becomes larger to be greater than \( \hat{N} \), strategic voters start to ignore their signals entirely when \( \theta \) is low, rather than keep a mixed strategy as in Theorem 2. Consider any \( N > \hat{N} \), at pivotality, there are \((2q - 1)(N - 1)\) more votes for \( A \) than for \( R \). Among these \((2q - 1)(N - 1)\) votes, a certain proportion of them is made by non-strategic voters who receive signal \( w \), and this proportion is increasing in \( \theta \).\(^{11}\) The higher \( \theta \) is, the more signal \( w \) are received by voters and then revealed by pivotality. More signal \( w \) revealed the richer information favoring state \( W \), and then stronger motivation for strategic voters to put less weight on their own signals. When the information is rich enough, strategic voters put no weight on their own signals, that is, they ignore their own signals. The following corollary summarize this intuition.

**Corollary 1** Consider any given \( \{N, q, p, \theta, \alpha\} \) with \( q, p, \alpha \in (\frac{1}{2}, 1) \), \( \theta \in (0, 1) \) such that a unique symmetric Nash equilibrium exists. Then in this Nash equilibrium, \( \sigma_q^* (nw) \) is decreasing in \( \theta \).

If the unique Nash equilibrium is of Type-I, then \( \sigma_q^* (nw) = \frac{1}{2} \cdot \sigma_{qF}^* (nw) \) is strictly decreasing with \( \theta \), as \( \sigma_{qF}^* (nw) \) is independent of \( \theta \). While when the unique Nash equilibrium is of Type-H, \( \sigma_q^* (nw) = 1 \), which does not change with \( \theta \).

### 6 Information Aggregation

Group decisions are considered to be better than decisions made by an individual in aggregation information. So it is natural for us to look how strategic

\(^{10}\) Notice that though the assumption \( \theta < \theta_{np} \) does not show explicitly in Theorem 3, we have to have it for Theorem 3.2, as we need \( \hat{N} \) to be finite.

\(^{11}\) By the Law of Large Numbers, the proportion is \((1 - \theta)p \) in state \( W \) and \((1 - \theta)(1 - p) \) in state \( NW \), given the population being large.
heterogeneity affects information aggregation. Since I focus on the influence on information aggregation imposed by strategic heterogeneity, I still adopt the fully-strategic model as the reference. Numerically, I compare the probabilities that mistakes happen, $Pr(\text{AP}|\text{NW})$ and $Pr(\text{RE}|W)$, in the model with strategic heterogeneity, to the corresponding probabilities, $Pr^*(\text{AP}|\text{NW})$ and $Pr^*(\text{RE}|W)$ in the fully-strategic model.

In the model with strategic heterogeneity, for any given $q$ and $N$, in equilibrium, $Pr^q(\text{AP}|\text{NW})$ and $Pr^q(\text{RE}|W)$ are determined by $\beta^*_{WN}$ and $\beta^*_{NWN}$. Recall that

\[
\beta^*_{WN} = (1 - \theta)Pr(w|W) + \theta \cdot \gamma^*_{WN}
\]

\[
\beta^*_{NWN} = (1 - \theta)Pr(w|NW) + \theta \cdot \gamma^*_{NWN}
\]

are the ex ante probabilities that a single vote ends in $A$ in state $W$ and $NW$, respectively. The first term of each $\beta^*_{SN}$, is the vote for $A$ made by a non-strategic voter who receives signal $A$ in state $S \in \{W, NW\}$. As $Pr(w|S)$ is the probability that a non-strategic votes $A$ in state $S$, which directly reflects signal $w$ sent by state $S$, the first term of $\beta^*_{SN}$ reflects how non-strategic voters’ signals directly affect the ex ante probability that a single vote ends in $A$. This influence on information aggregation imposed by strategic heterogeneity through non-strategic voters’ action the direct information effect.

The second term of $\beta^*_{SN}$, is the probability that a strategic voter votes $A$, given state $S \in \{W, NW\}$. As discussed earlier, $\gamma^*_{SN}$ pushed up by strategic heterogeneity through the “truthful” actions of non-straitening voters, which implies an indirect influence on information aggregation caused by strategic heterogeneity. That is why in early context, the effect of strategic heterogeneity on strategic voters’ behaviors is called indirect information effect.

As

\[
\gamma^*_{WN} = p + (1 - p)\sigma^*_{N}(nw) > p = Pr(w|W)
\]

\[
\gamma^*_{NWN} = (1 - p) + p\sigma^*_{N}(nw) > 1 - p = Pr(w|NW)
\]

in the unique Nash equilibrium for large $N$, we know that the indirect information effect and the direct information effect go in opposite directions. How strong the counteraction between them is relies mainly on the level of $\theta$, and also on the level of $q$, as shown in the following theorems.

For any given $\{N, q, p, \theta, \alpha\}$ with $q, p, \alpha \in (\frac{1}{2}, 1)$, $\theta \in (0, 1)$, define $\Theta_{q-} = \{\theta : \theta < \theta^*_{q}\}$ and $\Theta_{q+} = \{\theta : \theta > \theta^*_{q}\}$. 

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Proposition 3 Consider \( \{N,q,p,\theta,\alpha\} \) with \( q,p,\alpha \in \left( \frac{1}{2},1 \right), \theta \in (0,1) \). For given \( q \), define \( \Theta_{q^+} = \{ \theta : \theta > \theta_{qp} \} \), then for \( N > \hat{N}^* \), in the unique symmetric Nash equilibrium, i.e., the Type I Nash equilibrium, the probabilities that mistakes happen, \( \Pr_N^q(AP|NW) \) and \( \Pr_N^q(RE|W) \), are independent of \( \theta \in \Theta_{q^+} \).

Theorem 2 shows that for any \( \theta \in \Theta_{q^+} \), when \( N > \hat{N}^* \), the I strategy \( (\sigma_{q^*}^N(w),\sigma_{q^*}^N(nw)) = (1, \frac{1}{\theta} \cdot \sigma_{q^*}^N(nw)) \) is the unique symmetric Nash equilibrium strategy for strategic voters. Given this strategy, we have

\[
\gamma_{w_N}^{q^*} = p + (1-p) \frac{1}{\theta} \cdot \sigma_{q^*}^N(nw)
\]

and

\[
\gamma_{NW_N}^{q^*} = (1-p) + p \cdot \frac{1}{\theta} \cdot \sigma_{q^*}^N(nw),
\]

then

\[
\beta_{w_N}^{q^*} = (1-\theta)\Pr(w|W) + \theta \cdot \gamma_{w_N}^{q^*} = p + (1-p) \cdot \sigma_{q^*}^N(nw)
\]

and

\[
\beta_{NW_N}^{q^*} = (1-\theta)\Pr(w|NW) + \theta \cdot \gamma_{NW_N}^{q^*} = (1-p) + p \cdot \sigma_{q^*}^N(nw).
\]

As by Proposition 1 we know that \( \sigma_{q^*}^N(nw) \) is independent of \( \theta \), then the probabilities that mistakes happen,

\[
\Pr(\text{AP}|NW) = \sum_{m=qN}^{N} \binom{N}{m} \left( \frac{\beta_{NW_N}^{q_N}}{1-\beta_{NW_N}^{q_N}} \right)^m \left( 1-\beta_{NW_N}^{q_N} \right)^{N-m}
\]

and

\[
\Pr(\text{RE}|W) = 1 - \sum_{m=qN}^{N} \binom{N}{m} \left( \frac{\beta_{w_N}^{q_N}}{1-\beta_{w_N}^{q_N}} \right)^m \left( 1-\beta_{w_N}^{q_N} \right)^{N-m}
\]

are also independent of \( \theta \).

Proposition 4 Consider \( \{N,q,p,\theta,\alpha\} \) with \( q,p,\alpha \in \left( \frac{1}{2},1 \right), \theta \in (0,1) \). For \( N > \hat{N} \), in the unique symmetric Nash equilibrium, i.e., the Type II Nash equilibrium,

\[
(4.1) \quad \Pr_N^q(\text{AP}|NW), \text{ the probability that the policy is approved by mistake, strictly increases with } \theta \in \Theta_{q^+};
\]

\[
(4.2) \quad \Pr_N^q(\text{RE}|W), \text{ the probability that the policy is rejected by mistake, strictly decreases with } \theta \in \Theta_{q^+}.
\]

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In the Type H Nash equilibria in Proposition 4, we have

\[ \beta^q_{WN} = 1 - (1 - \theta)(1 - p) \]

and

\[ \beta^q_{NWN} = 1 - \theta(1 - P). \]

Then an increase in \( \theta \) causes \( \beta^q_{WN} \) to go up and \( \beta^q_{NWN} \) to fall. Intuitively, the ex ante probability that the policy is approved in state \( S \in \{W, NW\} \), \( Pr(AP|S) \), is increasing in \( \beta^q_{WN} \).\(^{12}\) Then an increase of \( \theta \) causes \( Pr(AP|NW) \) and \( Pr(AP|W) \) to increase, the latter of which implies a lower \( Pr(RE|W) \).

By Theorem 2 and Theorem 3, we know that in the model with strategic heterogeneity, the indirect information effect causes strategic voters to over-vote \( A \), which pushes \( \gamma^q_{WN} \) and \( \gamma^q_{NWN} \) higher. As in any state \( S \in \{W, NW\} \), the probability that a non-strategic voter votes \( A \), \( Pr(w|S) \), is always lower than \( \gamma^q_{WN} \); the probability that a strategic voter votes \( A \), the manipulation power coming from strategic voters’ over-voting for \( A \) is counteracted by non-strategic voters’ relatively lower tendency of voting \( A \), that is, the indirect information effect is counteracted by the direct information effect. Which information effect determines the different results in Proposition 3 and Proposition 4.

When \( \theta \) is relatively high, higher than \( \theta_{qp} \), pivotality is rich enough in information revelation, which leads to a relative weaker indirect information effect. The information effect is so weak that it is offset by the direct information effect, thus then ex ante probabilities that wrong collective decisions are made are not affected by the level of \( \theta \).

On the other hand, if \( \theta \) drops further to be strictly below \( \theta_{qp} \), the richness of pivotality in information revelation is so high that strategic voters ignore their own private signals entirely. Though the “swing voter’s curse” is pushed to its limit, the indirect information effect is dominated by the direct information effect. This is because these two information effects are weighted by the prior probabilities of voters’ types.

**Proposition 5** For given \( \{N, q, p, \alpha\} \) with \( q, p, \alpha \in (\frac{1}{2}, 1) \), consider \( \theta_1 \in \Theta_{q^+} \) and \( \theta_2 \in \Theta_{q^-} \). Then for any \( N > N \), in the unique symmetric Nash equilibrium,

\[ \beta^q_{WN}|e=e_1 > \beta^q_{WN}|e=e_2 \quad \text{and} \quad \beta^q_{NWN}|e=e_1 > \beta^q_{NWN}|e=e_2. \]

Proposition 5 shows that, though the swing voter’s curse is exacerbated by the drop in \( \theta \) through the indirect information effect in the sense that the manipulation power pro \( A \) from strategic voters’ actions reaches its limit, the influence

\(^{12}\)Actually, this intuition is true if \( \beta^q_{WN} > \frac{1}{2} \). I ignore this condition since both \( \beta^q_{WN} \) and \( \beta^q_{NWN} \) satisfies this requirement.
of the indirect information effect on the prior probabilities that the policy is approved in any given state $S \in \{W, NW\}$ is weakened, as $\theta$, which is an index of the share of strategic voters in the total population, decreases. Numerically, given $\theta < \theta_{qp}$, when $N > \hat{N}$, in the unique symmetric Nash equilibrium, we have

$$\beta^q_{WN} = 1 - (1 - \theta)(1 - p) \quad \text{and} \quad \beta^q_{NW_N} = 1 - (1 - \theta)p,$$

which shows that given any state $S \in \{W, NW\}$, the only votes against $A$ come from non-strategic voters who receive signal $nw$, which can be considered as another side of the direct information effect. The influence of these votes on $\beta^q_{SN}$, measured by $(1 - \theta)(1 - Pr(w|S))$, decreases in $\theta$, thus a drop in $\theta$ strengthens the direct information effect. When $\theta$ is lower enough as discussed above, the direct information effect becomes so strong that it exceeds the indirect information effect and then lowers the ex ante probability that the policy is approved in any given state $S \in \{W, NW\}$.

The change of $\theta$ does affect the values of probabilities that mistakes happen in collective decisions, but how it affects the effectiveness of information aggregation, measured by the asymptotical changes of $Pr(AP|NW)$ and $Pr(RE|W)$, is not straightforward from the above results. The following theorem shows that for most of time, strategic heterogeneity does not affect the effectiveness of information aggregation in equilibria.

**Theorem 4** Consider $\{N, q, p, \theta, \alpha\}$ with $q, p, \alpha \in (\frac{1}{2}, 1)$, $\theta \in (0, 1)$. For $N > \hat{N}$, in the unique symmetric Nash equilibrium,

(4.1) if $\theta > \frac{q - p}{1 - p}$, information is aggregated effectively;

(4.2) if $\theta \leq \frac{q - p}{1 - p}$, information is not aggregated effectively.

The influence of strategic heterogeneity on strategic voters’ behaviors and information aggregation is summarized in Figure 1.\textsuperscript{13}

## 7 Informationally Effective $q$-Rules

Consider a social planner, for example, a president of a board, who needs to find an “optimal” rule for collective decision for a group with strategically heterogeneous population. The key requirement for such rule is informational effectiveness, that is, whether the probabilities of wrong outcomes diminish to 0 when voter population goes to infinity.

\textsuperscript{13}$\theta_{qp} > \frac{q - p}{1 - p}$ holds for both $q > p$ and $q \leq p$. WOLG I only show the case of $q > p$ in Figure 1.
Figure 1: The Influences of Strategic Heterogeneity on Strategic Voters’ Behaviors and Information Aggregation

Before looking at informationally effective $q$-rules, we need to check how the level of $q$ affects voters’ behaviors when their population is strategically heterogeneous.

**Lemma 6** For any given $\{p, \theta\}$ with $p \in \left(\frac{1}{2}, 1\right)$ and $\theta \in (0, 1)$, there exists a unique $q \in \left(\frac{1}{2}, 1\right)$, denoted as $q_{p, \theta}$, such that

(6.1) if $q > q_{p, \theta}$, \[ \left(1 - \frac{1}{q}\right)^{1 - \theta} \frac{q^{1 - \theta}}{p^{1 - (1 - \theta)(1 - p)}} > \frac{1 - (1 - \theta)p}{1 - (1 - \theta)(1 - p)}; \]

(6.2) if $q < q_{p, \theta}$, \[ \left(1 - \frac{1}{q}\right)^{1 - \theta} \frac{q^{1 - \theta}}{p^{1 - (1 - \theta)(1 - p)}} < \frac{1 - (1 - \theta)p}{1 - (1 - \theta)(1 - p)}. \]

Then $q_{p, \theta} \in (1 - (1 - \theta)p, 1 - (1 - \theta)(1 - p))$. 

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Proposition 6 Consider any given \( \{q, p, \theta, \alpha\} \) with \( q, p, \alpha \in (\frac{1}{2}, 1) \) and \( \theta \in (0, 1) \). If \( q < q_{p \theta} \), for any \( N > \hat{N}^* \), there exists a unique symmetric Nash equilibrium, in which the strategic voters adopt I strategy \((\sigma^q_N(w), \sigma^q_N(nw)) = (1, \frac{1}{2} \cdot \sigma^F_N)\). That is, Type I Nash equilibrium is the unique symmetric Nash equilibrium when \( N \) is large.

Proposition 7 Consider any given \( \{N, q, p, \theta, \alpha\} \) with \( q, p, \alpha \in (\frac{1}{2}, 1) \), \( \theta \in (0, 1) \). If \( q > q_{p \theta} \), then

\(|7.1| \) if \( \hat{N} > \hat{N}^* + 1 \), then for any \( N \) such that \( \hat{N}^* < N < \hat{N} \), the Type I Nash equilibrium, constructed by the I strategy \((\sigma^q_N(w), \sigma^q_N(nw)) = (1, \frac{1}{2} \cdot \sigma^F_N)\) is the unique symmetric Nash equilibrium\(^\text{14}\);

\(|7.2| \) for any \( N > \hat{N} \), Type H Nash equilibrium is the unique symmetric Nash equilibrium.\(^\text{15}\)

The intuition behind Proposition 6 and Proposition 7 also comes from the richness of information revealed by pivotality. As mentioned earlier, at pivotality, there are \((2q-1)(N-1)\) more votes for A than for R. For any given \( \theta \) and \( N \), a higher \( q \) implies potentially more signal \( w \), which brings strategic voters stronger belief on state \( W \), that is, stronger indirect information effect, thus strategic voters have higher tendency to vote A. When \( q \) is relatively very high, greater than \( q_{p \theta} \), the indirect information effect becomes so strong that strategic voters place no weight on their own signals.

Though the higher \( q \) is, the more strategic voters tend to over-vote A, how \( P^q_N(\text{AP}|S) \), the probability that the policy is approved in state \( S \in \{W, NW\} \), change with \( q \) is not trivial, since a higher \( q \) means more votes for A needed by the policy to be approved. Actually, the effectiveness of a \( q \)-rule is more affected by the non-strategic voters’ actions than by the strategic voters’, as shown in Theorem 5, though which we can see how strategic heterogeneity, more precisely, the presence of non-strategic voters, imposes a upper bound for the level of effective \( q \)-rules. Such intuition is represented by the following corollary.

Corollary 2 Consider any given \( \{N, q, p, \theta, \alpha\} \) with \( q, p, \alpha \in (\frac{1}{2}, 1) \), \( \theta \in (0, 1) \) such that a unique symmetric Nash equilibrium exists. Then in this Nash equilibrium, \( \sigma^q_N(nw) \) is increasing in \( q \).

\(^{14}\)Here \( \sigma^F_N(nw) = pB^q_N - (1-p)B^q_N \), the same as defined in Proposition 1.

\(^{15}\)Notice that though the assumption \( \theta < \theta_{p \theta} \) does not show explicitly in Theorem 3, we have to have it for Theorem 3.2, as we need \( \hat{N} \) to be finite.
Since $\sigma_N^q(nw)$ is strictly increasing with $q$, if the unique Nash equilibrium is of Type-I, then $\sigma_N^q(nw) = \frac{1}{\theta} \cdot \sigma_N^{F^*}(nw)$ is strictly increasing with $q$. While when the unique Nash equilibrium is of Type-H, $\sigma_N^q(nw) = 1$, which does not change with $q$.

**Theorem 5** Consider $\{N, q, p, \theta, \alpha\}$ with $q, p, \alpha \in (\frac{1}{2}, 1)$, $\theta \in (0, 1)$. Information is aggregated effectively if and only if $q < 1 - (1 - \theta)(1 - p)$, that is:

\[
\lim_{N \to \infty} \Pr_N^q(\text{AP}|\text{NW}) = 0 \quad \text{and} \quad \lim_{N \to \infty} \Pr_N^q(\text{RE}|\text{W}) = 0,
\]

if and only if $q < 1 - (1 - \theta)(1 - p)$. If $q > 1 - (1 - \theta)(1 - p)$,

\[
\lim_{N \to \infty} \Pr_N^q(\text{AP}|\text{NW}) = 0 \quad \text{and} \quad \lim_{N \to \infty} \Pr_N^q(\text{RE}|\text{W}) = 1.
\]

The implication of Theorem 5 is analogue to Theorem 4, in the sense that if there are less strategic voters, i.e., when $\theta$ is low, the less $q$-rules with $q \in (\frac{1}{2}, 1)$ aggregates information effectively, since the lower $\theta$ is, the lower the upper bound for effective $q$-rules, $1 - (1 - \theta)(1 - p)$, is. This result comes from the fact that given population $N$, for any $q$-rule, we just need $(1 - q)N$ votes for $R$ to reject the policy. By the Law of Large Numbers, we know that there almost surely are $(1 - \theta) \cdot Pr(nw|S) \cdot N$ votes for $R$ in state $\xi \in \{W, NW\}$ when $N$ is large, made by non-strategic voters who receive signal $nw$. If $\theta$ is relatively too high, or $q$ is relatively too low, we have $(1 - \theta) \cdot Pr(nw|S) \cdot N \geq (1 - q)N$, which means that the policy will be rejected in any state. Though a low $\theta$ or a high $q$ implies stronger indirect information effect on strategic voters behaviors, on the overall information aggregation level, summing up all voters’ actions, indirect information effect is weakened by low $\theta$ or high $q$, and then get buried up by the direct information effect.

Strategic voters’ behavior under different $q$-rules and the effectiveness of different $q$-rules are summarized in Figure 2.

### 8 Welfare Analysis

Since non-strategic voters’ “always informative voting” strategy is neither affected by $\theta$ nor by $q$, their welfare must be determined by some factors beyond this paper, so to see how social welfare is affected by strategic heterogeneity and voting rules, I look how strategic voters’ utilities change with $\theta$ and $q$.

\[16\text{Lemma 7 in the appendix.}\]
8.1 Influence of Strategic Heterogeneity

**Theorem 6** Consider any given \( \{q, p, \theta, \alpha\} \) with \( q, p, \alpha \in \left( \frac{1}{2}, 1 \right) \).

(6.1) If \( \theta \in \Theta_{q^+} \) and \( N > \hat{N}^* \), or \( \theta \in \Theta_{q^-} \) and \( \hat{N}^* < N < \hat{N} \), strategic voters’ utilities are independent of \( \theta \);

(6.2) If \( \theta \in \Theta_{q^-} \), then for any \( N > \hat{N} \), strategic voters’ utilities decrease with \( \theta \).

Notice that in both cases of (6.1) and (6.2), voting \( A \) always maximizes strategic voters’ expected utility, regardless what signals they receive\(^\text{17}\), to see how strategic heterogeneity affect strategic voters’ expected utilities, I can just look at the influence of \( \theta \) on \( EU(A|s, piv) \), a strategic voter’s expected utility of voting \( A \) when he receives signal \( s \in \{w, nw\} \).

Recall that the only endogenous factor that affects \( EU_N(A|s, piv) \) is \( Pr(AP|NW) \), the probability that the policy is approved in state \( NW \), and \( EU_N(A|s, piv) \) decreases with \( Pr(AP|NW) \).

By Proposition 3 we know that, for \( \theta \in \Theta_{q^+} \) and \( N > \hat{N}^* \), in the unique symmetric Nash equilibrium in which strategic voters adopt the Type-I strategy

\(^{17}\)In a mixed strategy, voting \( A \) and voting \( R \) bring the same expected utility to strategic voters, then the expected utility from adopting this strategy is the same as the expected utility from voting \( A \).
profile, \( Pr(\text{AP}|\text{NW}) \) is independent of \( \theta \), which implies that \( EU_{N}(A|s,\text{piv}) \) does not change with \( \theta \). The same reason applies to the case in which \( \theta \in \Theta_{q-} \) and \( \hat{N}^* < N < \hat{N} \).

When \( \theta \in \Theta_{q-} \), then for any \( N > \hat{N} \), in the unique symmetric Nash equilibrium, by Proposition 4, \( Pr(\text{AP}|\text{NW}) \) increases with \( \theta \), therefore \( EU_{N}(A|s,\text{piv}) \) decreases with \( \theta \).

8.2 Influence of Voting Rules

Theorem 7 Consider any given \( \{q, p, \theta, \alpha\} \) with \( q, p, \alpha \in (\frac{1}{2}, 1) \) and \( \theta \in (0, 1) \). Define \( Q_{\theta-} = (q_{\theta}, 1) \) and \( Q_{\theta+} = (\frac{1}{2}, q_{\theta}) \).

(7.1) If \( q \in Q_{\theta-} \) and \( N > \hat{N}^* \), or \( q \in Q_{\theta+} \) and \( \hat{N}^* < N < \hat{N} \), strategic voters' utilities are independent of \( \theta \);

(7.2) If \( q \in Q_{\alpha+} \), then for any \( N > \hat{N} \), strategic voters' utilities increase with \( \theta \).

In both (7.1) and (7.2), a higher \( q \) makes pivotality richer in the information favoring state \( W \), then strategic voters have higher tendency to vote \( A \), which pushes the probability that a single vote ends for \( A, \beta_{N}\), up, and also pushes \( Pr(\text{AP}|\text{NW}) \) up. Meanwhile, a higher \( q \) means more votes needed to support \( A \) at pivotality, for any given size of the voter population, which lowers \( Pr(\text{AP}|\text{NW}) \). Then an increase in \( \beta_{N}^{q} \) and an increase in \( q \) drag \( Pr(\text{AP}|\text{NW}) \) in opposite directions. In case (7.1), their influences cancel each other. Such offset keeps \( Pr(\text{AP}|\text{NW}) \) unchanged when \( q \) changes, then \( EU_{N}(A|s,\text{piv}) \) is not affected by the change of \( q \).

In case (7.2), \( \beta_{N}^{q} \) has been pushed to its limit, and does not change with \( q \), then the influence of the change of \( q \) on \( Pr(\text{AP}|\text{NW}) \) dominates. Such effect lowers \( Pr(\text{AP}|\text{NW}) \) when \( q \) increases, which increases \( EU_{N}(A|s,\text{piv}) \).

9 Voluntary Voting

The model discussed above focuses on mandatory voting. Now let’s turn to the voluntary voting case in which voters can abstain. To have insight into it, we need to consider two voting rules: the simple majority rule, in which \( q = \frac{1}{2} \), and a general voting rule with \( q > \frac{1}{2} \).

First of all, we should notice that actually there are two types of pivotalities in voluntary voting, in one of which a vote for \( A \) is pivotal and in the other case a vote for \( R \) is pivotal. A vote for \( A \) is pivotal implies that the policy is going to be approved only if the pivotal voter turns out to vote and vote \( A \), and if he vote \( R \) or abstain, the policy will be rejected. Similarly, A vote for \( R \) is pivotal implies that the policy is going to be rejected only if the pivotal
voter turns out to vote and vote R, and if he vote A or abstain, the policy will be approved. Call the former type as $Piv_A$ and the latter one as $Piv_C$. Since the number of votes on any alternative must be an integer, $Piv_A$ and $Piv_C$ do not exist at the same time but take turns to show up, and the frequencies of them turning-out depends on the ration $\frac{q}{1-q}$. More precisely, $Piv_R$ shows up $\frac{q}{1-q}$ times more frequently as $Piv_A$.\(^{18}\)

In mandatory voting model, we do not need to worry about such frequencies, as there is only one type of pivotality, $Piv_C$, in mandatory voting. That is why in earlier section we can assume that $qN$ is an integer. But in a voluntary model in which voters have the third choice, to abstain, such frequencies matter.

In voluntary voting under the simple majority rule, $Piv_A$ and $piv_C$ shows up at the same frequency, so abstention has equal influence on both sides, in the sense that abstention in pivotalities lead to $AP$ or $RE$ with equal probabilities, but under any other $q$-rule, abstention in pivotalities lead to $AP$ as $\frac{q}{1-q}$ times as frequently as to $RE$. Thus abstention has more manipulating power under a general $q$-rule than under a simple majority rule.

Furthermore, pivotalities have very different richness in information revelation under a general $q$-rule and the simple majority rule. Let’s look at the simple majority rule first, which can be considered as a special $q$-rule with $q = \frac{1}{2}$. Under this voting rule, turning out to vote and voting informatively can be a Nash equilibrium for strategic voters in certain circumstance in a costless voting model\(^ {19}\). The intuition for the existence of this Nash equilibrium comes from the special characteristic of simple majority rule: at pivotality, the difference between the numbers of votes pro $A$ and against it is just one, independent of the population of voters. This is different from general $q$-rule with $q \in (\frac{1}{2}, 1)$, the gap of the latter is determined by the population of voters and it is unbounded when the population increases to infinity.\(^ {20}\)

Such characteristic of the simple majority rule affects the strategic voters’ strategies in two aspects. First, pivotality is not very rich in information. At any pivotality, the numbers of votes pro and against $A$ are so close, the information revealed by pivotalities supporting the true state to be $W$ is not meaningful enough for a strategic voter to ignore his own signal. On the opposite, his privet signal becomes so comparatively richer in information that he tends to follow it. But for a general $q$-rule, the gap between the numbers of votes pro and against $A$ at pivotalities is large when the total population is large, which implies far more votes supporting $A$ than supporting $R$, in this sense, pivotalities become very rich in information revelation, and such information favors the possibility

\(^{18}\)See Xu (2013) for more details.

\(^{19}\)It depends on the relation between $q$ and $p$.

\(^{20}\)For given $q > \frac{1}{2}$, such gap is $\lfloor (2q - 1)N \rfloor$ or $\lfloor (2q - 1)N \rfloor + 1$, depending on whether the pivotal vote is pro $A$ or against it. Here $\lfloor (2q - 1)N \rfloor$ is the greatest integer that is lower than $(2q - 1)N$.  

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that the true state is \( W \). This brings strong motivation for strategic voters to vote \( A \).

Though with the richness in information revelation, I find that turning out and voting informatively is not a Nash equilibrium strategy in voluntary voting under general \( q \)-rule, it is very difficult to find out all the potential Nash equilibrium strategies in this case. Such difficulty is caused by the frequencies of \( Piv_A \) and \( Piv_C \), as there is no close form for all the conditional probabilities, \( Pr(S|s, Piv) \), where \( S\{NW, W\} \) represents for the true state and \( s \in \{nw, w\} \) stands for the signal, or at least I have not found the close forms, which keeps me away from going further.

Before we proceed to the concluding section, one important point needs to be clarified and emphasized before we proceed. Some people may have suspicion that \( \hat{N} \) and \( \hat{N}^* \) may be too high to be realistic, for example, if \( \hat{N} \) is greater than 7 billions, which is the total amount population of people on the earth, the results in this paper may not be meaningful. The following example can soothe the worry.

**Example 1** Consider the following sets of \( \{q, p, \theta, \alpha\} \).

\[(1.1) \quad \{q, p, \theta, \alpha\} = \{0.9, 0.8, 0.1, 0.9\}, \text{ then } \hat{N}^* = 2 \text{ and } \hat{N} = 5;\]
\[(1.2) \quad \{q, p, \theta, \alpha\} = \{0.9, 0.8, 0.5, 0.9\}, \text{ then } \hat{N}^* = 2 \text{ and } \hat{N} = 18;\]
\[(1.3) \quad \{q, p, \theta, \alpha\} = \{0.9, 0.8, 0.9, 0.9\}, \text{ then } \hat{N}^* = 2 \text{ and } \hat{N} = +\infty.\]

The example does not only show that we do not need to worry too much that \( \hat{N}^* \) is too tremendous to have realistic meaning, but also shows that the results in this paper may be more meaningful than they seem. Though most results in this paper require “large” populations, i.e., \( N > \hat{N}^* \), the critical cut point \( \hat{N}^* \) may not be so high that the results can be applied only to national referendum. Actually, \( \hat{N}^* \) can be of such small size that these results in this paper can be applied to collective decisions in a board, in congress, or even to jury trials, given certain ranges of parameters. This is also meaningful in finding a proper size of a group to make decisions. For example, if large size of a board means higher cost in making decisions for a company, which gives the company motivations to minimize the size of board, then the above conclusions gives hints on the lower bound of the size.

### 10 Conclusions and Extensions

This paper investigates what happens when non-strategic voters are introduced into a standard 2-states\( \times \)2-alternatives voting model with asymmetric information. Such strategic heterogeneity have both positive and negative influences
on information aggregation, through which it affects collective decisions. On one hand, the direct information effect brings stronger motivation for strategic voters to over-vote on approving the policy. On the other hand, the influence of the information effect on collective decision is offset or overwhelmed by the direct information effect. How strategic heterogeneity affects collective decision relies on the result of their counteraction.

There are several extensions worth future effort. One is to allow abstention into our mandatory voting model, as in real life, most voting system adopts volunteer voting. Besides the potential influence on information aggregation, strategic heterogeneity may also affects strategic voters’ decision in abstention, which is important in researches on turnout rate in the real world.

Another extension is to introduce heterogeneity of information into our model. It is natural to think that different people may have signals with different quality. For example, an economist may have better understanding of a fiscal policy than a sports man who has little training in economics, and such difference in understanding the policy gives these two people different quality in precision of signals.

Appendix

Proof. (Theorem 1) As the results applies to any $q \in (\frac{1}{2}, 1)$ and $N \geq 2$, I omit the superscript $q$ and the subscript $N$ in this proof.

Let $EU(T|s, piv)$ denote a strategic voter’s expected utility of taking action $T$ when he receives signal $s$, where $T \in \{A, R\}$ and $s \in \{w, nw\}$, conditioning on being pivotal. Then

$$EU(R|s, piv) = U(RE, W) \cdot Pr(W|s, piv) + U(RE, NW) \cdot Pr(NW|s, piv)$$

$$= -(1 - \alpha) \cdot Pr(W|s, piv) < 0,$$

and

$$EU(A|s, piv) = -\alpha \cdot Pr(NW|s, piv) < 0,$$

where $s \in \{w, nw\}$.

$\sigma(nw) > 0$ if and only if $EU(A|nw, piv) \geq EU(R|nw, piv)$, which is equivalent to

$$\frac{Pr(NW|nw, piv)}{Pr(W|nw, piv)} \leq \frac{1 - \alpha}{\alpha},$$
then by
\[
\frac{Pr(NW|nw, piv)}{Pr(W|nw, piv)} = \frac{Pr(NW, nw, piv)}{Pr(W, nw, piv)} = \frac{Pr(NW) \cdot Pr(piv|NW) \cdot Pr(nw|piv, NW)}{Pr(W) \cdot Pr(piv|W) \cdot Pr(nw|piv, W)} = \frac{Pr(piv|NW) \cdot Pr(nw|NW)}{Pr(piv|W) \cdot Pr(nw|W)} = \frac{p}{1 - p} \cdot \frac{Pr(piv|NW)}{Pr(piv|W)},
\]
as \(Pr(NW) = Pr(W)\) and signals follow a i.i.d distribution, we have
\[
\frac{Pr(piv|NW)}{Pr(piv|W)} \leq \frac{(1 - \alpha)(1 - p)}{\alpha p}.
\]

Then
\[
\frac{EU(A|w, piv)}{EU(R|w, piv)} = \frac{\alpha}{1 - \alpha} \cdot \frac{Pr(NW|w, piv)}{Pr(W|w, piv)} = \frac{\alpha}{1 - \alpha} \cdot \frac{1 - p \cdot Pr(piv|NW)}{p \cdot Pr(piv|W)} \leq \frac{\alpha}{1 - \alpha} \cdot \frac{1 - p}{p} \cdot \frac{(1 - \alpha)(1 - p)}{\alpha p} < 1,
\]
and by \(EU(R|w, piv) < 0\) and \(EU(A|w, piv) < 0\) we have
\[
EU(A|w, piv) > EU(R|w, piv),
\]
which implies that the optimal response for a strategic voter who receives signal \(w\) is to vote \(A\) over \(R\), that is, \(\sigma(w) = 1\). We have the first result.

As \(\sigma(w) < 1\) if and only if \(EU(A|w, piv) \leq EU(R|w, piv)\), which is equivalent to
\[
\frac{Pr(piv|NW)}{Pr(piv|W)} \geq \frac{1 - \alpha}{\alpha} \cdot \frac{p}{1 - p}.
\]
Then a strategic voter who receives signal \(nw\) must have
\[
\frac{EU(A|nw, piv)}{EU(R|nw, piv)} = \frac{\alpha}{1 - \alpha} \cdot \frac{Pr(NW|nw, piv)}{Pr(W|nw, piv)} = \frac{\alpha}{1 - \alpha} \cdot \frac{p}{1 - p} \cdot \frac{Pr(piv|NW)}{Pr(piv|W)} \geq \frac{\alpha}{1 - \alpha} \cdot \frac{p}{1 - p} \cdot \frac{1 - \alpha}{\alpha} \cdot \frac{p}{1 - p} = \left(\frac{p}{1 - p}\right)^2 > 1,
\]
which implies
\[
EU(A|nw, piv) < EU(R|nw, piv),
\]
as $EU(A|nw, piv) < 0$ and $EU(R|nw, piv) < 0$, then strategic voters who receive signal $nw$ strictly prefer $R$ to $A$, therefore $\sigma(nw) = 0$. ■

**Proof. (Lemma 1)** Easy to see that the function $f(x) = \frac{1}{1 + \left(\frac{1-p}{p}\right)^{4q-4}x}$ is continuous and strictly increasing with $x \geq 2$.

Since $f(2) = \frac{1}{1 + \left(\frac{1-p}{p}\right)^{4q-4}} > 1 > \frac{1-\alpha}{\alpha}$ as $\alpha, p \in (\frac{1}{2}, 1)$, we have $f(2) < \alpha$. As $\lim_{x \to \infty} f(x) = 1 > \alpha$, by the continuity and monotonicity of $f(x)$ in $(2, \infty)$ we know that there exists $x^* \in (2, \infty)$ such that $f(x^*) = \alpha$ and such $x^*$ is unique.

Take $\hat{N}^* = [x^*]$, i.e., $\hat{N}^*$ is the greatest integer that is lower than $x^*$. Then $\hat{N}^*$ is unique. Therefore

$$
\frac{1}{1 + \left(\frac{1-p}{p}\right)^{4q-4}(\hat{N}^*+1)-2} = f(\hat{N}^* + 1) > f(x^*) = \alpha

\geq f(\hat{N}^*)

= \frac{1}{1 + \left(\frac{1-p}{p}\right)^{4q-4}(\hat{N}^*+1)-2,}

$$
as $\hat{N}^* + 1 > x^* \geq \hat{N}^*$. ■

**Proof. (Proposition 1)** By Proposition 1 we know that, for any $N > \hat{N}^*$ Type I Nash equilibrium exists as the unique symmetric responsive Nash equilibrium, which implies that strategies of Type II or M are Nash equilibrium strategies for strategic voters.

Consider the I strategy, with which every strategic voter votes $A$ regardless of their signals. For any $N > \frac{1}{1-q}$, a single strategic voter knows that he is not pivotal if all other $N - 1$ voters are voting $A$, then he is indifferent between voting $R$ and $A$. Thus he has no motivation to deviate from voting $A$, therefore I strategy is a Nash equilibrium strategy for strategic voters when $N > \max\{\hat{N}^*, \frac{1}{1-q}\}$.

Now look at the O strategy. If a strategic voter believes that all other $N - 1$ voters are voting $R$, then for any $N \geq 2$, this voter is not pivotal, therefore he is indifferent between voting $R$ and $A$ and O strategy is a symmetric Nash equilibrium strategy. ■

**Proof. (Proposition 2)** As for any given size of population, in the Type H Nash equilibrium, the probability of approving the policy by mistake in state $NW$ is 1, while the probability that the policy is wrongly rejected in state $W$ is 1 in the Type O Nash equilibrium, information is not aggregated effective in
these two Nash equilibria. ■

**Proof. (Lemma 2)** By solving \((\frac{1-p}{p})^{\frac{1-q}{q}} = \frac{1-(1-\theta)p}{1-(1-\theta)(1-p)}\), we have \(\theta = \frac{p\cdot(\frac{1-p}{p})^{\frac{1-q}{q}} - (1-p)}{p - (1-p)(\frac{1-p}{p})^{\frac{1-q}{q}}}\), then for any given \(q, p \in (\frac{1}{2}, 1)\), define \(\theta_{qp} = \frac{p\cdot(\frac{1-p}{p})^{\frac{1-q}{q}} - (1-p)}{p - (1-p)(\frac{1-p}{p})^{\frac{1-q}{q}}}\).

For any \(\theta < \theta_{qp}\), we have \(\theta < \frac{p\cdot(\frac{1-p}{p})^{\frac{1-q}{q}} - (1-p)}{p - (1-p)(\frac{1-p}{p})^{\frac{1-q}{q}}}\). By rearranging this inequality we have \((\frac{1-p}{p})^{\frac{1-q}{q}} > \frac{1-(1-\theta)p}{1-(1-\theta)(1-p)}\).

Similarly, for any \(\theta > \theta_{qp}\), which is equivalent to \(\theta > \frac{p\cdot(\frac{1-p}{p})^{\frac{1-q}{q}} - (1-p)}{p - (1-p)(\frac{1-p}{p})^{\frac{1-q}{q}}}\), we have \((\frac{1-p}{p})^{\frac{1-q}{q}} < \frac{1-(1-\theta)p}{1-(1-\theta)(1-p)}\). ■

**Proof. (Lemma 3)** If \(q \leq p\), \(\frac{q-p}{1-p} \leq 0 < \theta_{qp}\).

If \(q > p\), suppose that \(\theta_{qp} \leq \frac{q-p}{1-p}\), then by the definition of \(\theta_{qp}\) we have
\[
\left(\frac{1-p}{p}\right)^{\frac{1-q}{q}} \leq \frac{1-(1-\frac{q-p}{1-p})p}{1-(1-\frac{q-p}{1-p})(1-p)},
\]
which is equivalent to
\[
\left(\frac{1-p}{p}\right)^{\frac{1}{q}} + \frac{2p-1}{p} \cdot \frac{1}{q} \leq 1.
\]
Yet as \(q < 1\) we have
\[
\left(\frac{1-p}{p}\right)^{\frac{1}{q}} + \frac{2p-1}{p} \cdot \frac{1}{q} > \frac{1}{p} + \frac{2p-1}{p} = 1,
\]
an contradiction to the above inequality. Thus \(\theta_{qp} > \frac{q-p}{1-p}\). ■

**Lemma 7** Given \(\{q, p, \theta, \alpha\}\) with \(q, p, \alpha \in (\frac{1}{2}, 1)\) and \(\theta \in (0, 1)\), both \(B_N^q\) and \(\sigma^{qF^*}(nw) = \frac{pB_N^q - (1-p)}{p - (1-p)B_N^q}\) are strictly increasing with \(N\) and \(q\), and \(B_N^q < \left(\frac{1-p}{p}\right)^{\frac{1-q}{q}}, \forall N \in \mathbb{N}\).

**Proof.** The function \(y(q, x) = \left(\frac{1-q}{\alpha}\right)^{\frac{1}{1-q}} \cdot \left(\frac{1-p}{p}\right)^{\frac{x}{1-q}-1}\) is strictly increasing with \(q, x \in \mathbb{R}_+\), thus \(B_N^q = y(N, q)\) is strictly increasing with \(N\) and \(q\), and \(B_N^q < \lim_{N \to \infty} B_N^q = \left(\frac{1-p}{p}\right)^{\frac{1-q}{q}}, \forall N\).
As \( \frac{\partial \sigma^*}{\partial B_N} = \frac{2p-1}{p-1-p/B_N^q} \) > 0, \( \sigma^* \) is strictly increasing with \( B_N^q \). Thus \( \sigma^* \) strictly increases with \( N \) and \( q \). ■

**Proof. (Lemma 4)** \( \sigma^*_{N}(nw) \geq \theta \) is equivalent to \( B_N^q \geq \frac{1-(1-\theta)p}{1-(1-\theta)(1-p)} \), then

\[
B_N^q = \left( \frac{1-\alpha}{\alpha} \right)^{\frac{1}{2q-1}} \cdot \left( \frac{1-p}{p} \right)^{\frac{2}{2q-1}}.
\]

Thus

\[
\frac{1-p}{p} < \frac{1-(1-\theta)p}{1-(1-\theta)(1-p)}.
\]

we have \( \sigma^*_{N}(nw) < \theta \).

By Lemma 7 we know that \( B_N^q \) is strictly increasing with \( N \), then there exists some finite \( N \), denoted by \( \hat{N} \), such that \( B_{\hat{N}}^q \geq \frac{1-(1-\theta)p}{1-(1-\theta)(1-p)} \) if and only if

\[
\lim_{N \to \infty} B_N^q = \left( \frac{1-\alpha}{\alpha} \right)^{\frac{1}{2q-1}} \cdot \frac{1-p}{p} > \frac{1-(1-\theta)p}{1-(1-\theta)(1-p)} \text{, and by Lemma 2 we know that the latter holds if and only if } \theta < \theta_{\alpha} \text{.}
\]

Since \( \sigma^*_{N}(nw) \) is strictly increasing with \( N \), then when \( \theta < \theta_{\alpha} \), there exists a unique \( N \), denoted as \( \hat{N} \), that is finite, such that \( B_{\hat{N}}^q \geq \frac{1-(1-\theta)p}{1-(1-\theta)(1-p)} > B_N^q \), then \( \sigma^*_{\hat{N}}(nw) \geq \theta > \sigma^*_{N-1}(nw) \).

When \( \theta > \theta_{\alpha} \), we know that for any \( N \), \( \sigma^*_{N}(nw) < \theta \), since \( B_N^q < \frac{1-(1-\theta)p}{1-(1-\theta)(1-p)} \). Let \( \hat{N} = \infty \), which is also unique. ■

**Corollary 3** For any given \{q, p, \theta, \alpha\} with \( q, p, \alpha \in \left( \frac{1}{2}, 1 \right) \) and \( \theta \in (0, 1) \),

\( B_N^q > \frac{1-(1-\theta)p}{1-(1-\theta)(1-p)} \) if and only if \( N > \hat{N} \).

This corollary comes straightforward from the definition of \( \hat{N} \).

**Corollary 4** For any given \{q, p, \theta, \alpha\} with \( q, p, \alpha \in \left( \frac{1}{2}, 1 \right) \) and \( \theta \in (0, 1) \):

(4.1) \( \left( \frac{1}{p} \right)^{(2q-1)N-2} < \frac{1-\alpha}{\alpha}, \forall N > \hat{N}^* \);

(4.2) \( \left( \frac{1}{p} \right)^{(2q-1)N-2} > \frac{1-\alpha}{\alpha}, \forall N < \hat{N}^* \).

This corollary is straightforward from Lemma 2.
Lemma 8 For any \( \{N, q, p, \theta, \alpha\} \) with \( N > \hat{N} \), \( q, p, \alpha \in (\frac{1}{2}, 1) \) and \( \theta \in (0, 1) \),

\( (8.1) \quad \sigma_{N}^{F}(nw) > \theta \) if and only if \( \theta < \theta_{qp} \);

\( (8.2) \quad \sigma_{N}^{F}(nw) < \theta \) if and only if \( \theta > \theta_{qp} \).

Proof. (Theorem 2) First, we show that O, II, and M strategies cannot be Nash equilibrium strategies for strategic voters when the population of voters is greater than \( \hat{N} \).

To see this, notice that these three types of strategies share the following characteristics: \( \sigma_{N}^{q}(w) \leq 1 \) and \( \sigma_{N}^{q}(nw) = 0 \). Consider a strategic voter who believes that all other strategic voters are using the same strategy \( (\sigma_{N}^{q}(w), \sigma_{N}^{q}(nw)) \) with \( \sigma_{N}^{q}(w) \leq 1 \) and \( \sigma_{N}^{q}(nw) = 0 \). Then in his belief, we know

\[
\beta_{N}^{q} = (1 - \theta)p + \theta p \sigma_{N}^{q}(w) \leq p,
\]

and

\[
\beta_{N}^{q}(nw) = (1 - \theta)(1 - p) + \theta (1 - p) \sigma_{N}^{q}(w) \leq 1 - p.
\]

Notice that

\[
\beta_{N}^{q}(nw) = \frac{1 - \theta}{p} \beta_{N}^{q}.
\]

As the function \( \eta(x) = \frac{1 - \theta}{1 - p} x \) is strictly increasing with \( x \in (0, 1) \), we have

\[
\frac{1 - \beta_{N}^{q}(nw)}{1 - \beta_{N}^{q}} = \frac{1 - \theta}{1 - p} \beta_{N}^{q} \leq \frac{1 - \theta}{1 - p} p = \frac{p}{1 - p}.
\]

Then if this strategic voter receives signal \( nw \), by

\[
\frac{EU_{N}(A|nw, piv)}{EU_{N}(R|nw, piv)} = \alpha \cdot \frac{p}{1 - \alpha} \cdot \frac{Pr_{N}(piv|NW)}{Pr_{N}(piv|W)}
\]

\[
= \alpha \cdot \frac{p}{1 - \alpha} \cdot \frac{\beta_{N}^{q}(nw)}{\beta_{N}^{q}} \cdot \frac{(\frac{N - 1}{qN - 1})^{qN - 1} \cdot (1 - \beta_{N}^{q}(nw))}{(1 - \theta)(1 - p) \sigma_{N}^{q}(w)} \leq \alpha \cdot \frac{p}{1 - \alpha} \cdot \frac{\beta_{N}^{q}(nw)}{1 - \beta_{N}^{q}(nw)} \cdot \frac{(1 - p)}{p} \cdot \frac{(1 - \theta)(1 - p) \sigma_{N}^{q}(w)}{(1 - \theta) \sigma_{N}^{q}(nw)} \leq \frac{\beta_{N}^{q}(nw)}{\beta_{N}^{q}} \cdot \frac{(1 - p)}{p} \cdot \frac{(\theta^{1 - \theta})}{(1 - \theta)(1 - p) \sigma_{N}^{q}(w)} \leq \frac{\alpha}{1 - \alpha} \cdot \frac{(1 - \theta)}{p} \cdot \frac{(\theta^{1 - \theta})}{(1 - \theta)(1 - p) \sigma_{N}^{q}(w)} \leq 1,
\]

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since by Corollary 4 $N > \hat{N}^*$, \( \left( \frac{1-p}{p} \right)^{(2q-1)N-2} \leq \left( \frac{1-p}{p} \right)^{(2q-1)(\hat{N}^*+1)-2} < \frac{1-\alpha}{\alpha} \). Thus $EU_N(A|nw, piv) > EU_N(R|nw, piv)$ as $EU_N(A|nw, piv) < 0$ and $EU_N(R|nw, piv) < 0$, which implies that this strategic voter’s best response is to vote $A$, i.e., $\sigma_N^q(nw) = 1$, then any symmetric strategy with $\sigma_N^q(nw) = 0$ cannot be a Nash equilibrium for strategic voters. In other words, strategies of type O, II, and M are not Nash equilibrium strategies for strategic voters.

Next I show that $H$ strategy is not a Nash equilibrium strategy.

Consider a strategic voter who believes that all other strategic voters votes $A$ regardless of their own private signals. Then in his belief,

\[
\beta_{NW_N}^q = (1 - \theta)(1 - p) + \theta = 1 - (1 - \theta)(1 - p)
\]

and

\[
\beta_{WN_N}^q = (1 - \theta)p + \theta = 1 - (1 - \theta)p,
\]

thus

\[
\frac{1 - \beta_{NW_N}^q}{1 - \beta_{WN_N}^q} = \frac{p}{1 - p}.
\]

As $\theta > \theta_p$, we have \( \left( \frac{1-p}{p} \right)^{\frac{1-\theta}{1-\theta_p}} < \frac{1 - (1 - \theta)p}{1 - (1 - \theta_p)(1 - p)} \). Then by Lemma 7 we have

\[
B_N^q = \left[ \left( \frac{1-\alpha}{\alpha} \right) \cdot \left( \frac{1-p}{p} \right)^{(1-q)N+1} \right]^{\frac{1-\theta}{1-\theta_p}} < \left( \frac{1-p}{p} \right)^{\frac{1-\theta}{1-\theta_p}} < \frac{1 - (1 - \theta)p}{1 - (1 - \theta_p)(1 - p)}, \forall N \geq 2,
\]

thus

\[
\left[ \frac{1 - (1 - \theta)p}{1 - (1 - \theta_p)(1 - p)} \right]^{qN-1} > \frac{1-\alpha}{\alpha} \cdot \left( \frac{1-p}{p} \right)^{(1-q)N+1}.
\]

Then if this strategic voter receives signal $nw$, he has

\[
\frac{EU_N(A|nw, piv)}{EU_N(R|nw, piv)} = \frac{\alpha}{1 - \alpha} \cdot \frac{p}{1 - p} \cdot \left( \frac{\beta_{NW_N}^q}{\beta_{WN_N}^q} \right)^{qN-1} \cdot \left( \frac{1 - \beta_{NW_N}^q}{1 - \beta_{WN_N}^q} \right)^{(1-q)N} = \frac{\alpha}{1 - \alpha} \cdot \frac{p}{1 - p} \cdot \frac{1 - (1 - \theta)p}{1 - (1 - \theta_p)(1 - p)} \cdot \left( \frac{1-p}{p} \right)^{(1-q)N} > \frac{\alpha}{1 - \alpha} \cdot \frac{p}{1 - p} \cdot \frac{1 - \alpha}{\alpha} \cdot \frac{1-p}{p} \cdot \left( \frac{1-p}{p} \right)^{(1-q)N+1} = 1,
\]

thus $EU_N(A|nw, piv) < EU_N(R|nw, piv)$. Therefore this strategic voter’s best response when receiving signal $nw$ is to vote $R$, rather than votes $A$. Hence $H$ strategy is not a symmetric Nash equilibrium strategy.

Now we show that there exists a I strategy that is a symmetric Nash equilibrium strategy.
Consider a strategic voter who believes that all other strategic voters adopt I strategy $(\sigma^q_N(w), \sigma^q_N(nw))$ with $\sigma^q_N(w) = 1$ and $\sigma^q_N(nw) \in (0, 1)$. Then this I strategy constructs a Nash equilibrium if and only if this strategic voter has $EU^q_N(R|nw, piv) = EU^q_N(A|nw, piv)$ and $EU^q_N(R|w, piv) < EU^q_N(A|w, piv)$.

By Theorem 1 we know that once $EU^q_N(R|nw, piv) = EU^q_N(A|nw, piv)$, we must have $EU^q_N(R|w, piv) < EU^q_N(A|w, piv)$, so to look for the equilibrium strategy, I just need to solve $EU^q_N(R|nw, piv) = EU^q_N(A|nw, piv)$.

For this strategic voter,

$$\beta^q_{NW_N} = (1 - \theta)(1 - p) + \theta [(1 - p) + p\sigma^q_N(nw)] = 1 - [1 - \theta\sigma^q_N(nw)] p$$

and

$$\beta^q_{WN_N} = (1 - \theta)p + \theta [p + (1 - p)\sigma^q_N(nw)] = 1 - [1 - \theta\sigma^q_N(nw)] (1 - p),$$

then by solving

$$\frac{EU^q_N(R|nw, piv)}{EU^q_N(A|nw, piv)} = \frac{1 - \alpha}{\alpha} \cdot \frac{1 - p}{p} \cdot \left(\frac{\beta^q_{WN_N}}{\beta^q_{NW_N}}\right)^{qN-1} \cdot \left(\frac{1 - \beta^q_{WN_N}}{1 - \beta^q_{NW_N}}\right)^{(1-q)N}$$

$$= \frac{1 - \alpha}{\alpha} \cdot \frac{1 - p}{p} \cdot \left\{ \frac{1 - [1 - \theta\sigma^q_N(nw)] (1 - p)}{1 - [1 - \theta\sigma^q_N(nw)] p} \right\}^{qN-1} \cdot \left(\frac{1 - p}{p}\right)^{(1-q)N}$$

$$= 1,$$

we have

$$\sigma^q_N(nw) = \frac{1}{\theta} \cdot \frac{p\beta^q_{WN_N} - (1 - p)}{p - (1 - p)\beta^q_{WN_N}} = \frac{1}{\theta} \cdot \sigma^q_{F*}(nw),$$

where $B^q_N = (1 - \alpha) \frac{1}{\theta} \cdot \frac{(1 - p)}{p} \frac{1}{\theta} = \frac{1}{\theta} \cdot \sigma^q_{F*}(nw)$. As $\theta > \theta_{wp}$, we have $\sigma^q_{F*}(nw) < \theta$ by Lemma 4, which implies that the above $\sigma^q_N \in (0, 1)$.

From above we can see that I strategy $(\sigma^q_N(w), \sigma^q_N(nw)) = (1, \frac{1}{\theta} \cdot \sigma^q_N)$ is a symmetric Nash equilibrium strategy for strategic voters, therefore Type I Nash equilibrium exits. Also from above we can see that this Type I Nash equilibrium with $(\sigma^q_N(w), \sigma^q_N(nw))$ is the unique Type I Nash equilibrium when $\theta > \theta_{wp}$ and $N > N$. ■

**Proof. (Lemma 5)** By Corollary 3, we have $B^q_N > \frac{1 - (1 - \theta)p}{1 - (1 - \theta)(1 - p)} > B^q_{N - 1}$, which is equivalent to

$$\left[ \frac{1 - (1 - \theta)p}{1 - (1 - \theta)(1 - p)} \right]^{q(N-1) - 1} \cdot \left(\frac{1 - p}{p}\right)^{(q-1)(N-1) - 1} \geq \frac{1 - \alpha}{\alpha} > \left[ \frac{1 - (1 - \theta)p}{1 - (1 - \theta)(1 - p)} \right]^{qN - 1} \cdot \left(\frac{1 - p}{p}\right)^{(q-1)N - 1}.$$ 

By Corollary 4, we have
\[(1-p)^{(2q-1)\hat{N}^*-2} \geq \frac{1-\alpha}{\alpha} > \frac{(1-p)^{(2q-1)(\hat{N}^*+1)-2}}{N\ast - 2}.\]

Suppose that \(\hat{N}^* \geq \hat{N}\), as \(\frac{1-(1-\theta)p}{1-(1-\theta)(1-p)} > \frac{1-p}{p}\), we have

\[
\left[1 - \frac{(1-\theta)p}{1-(1-\theta)(1-p)}\right]^{q\hat{N}-1} \cdot \left(\frac{1-p}{p}\right)^{(q-1)\hat{N}-1} > \left(\frac{1-p}{p}\right)^{(2q-1)\hat{N}-2} \geq \frac{1-\alpha}{\alpha},
\]

which conflicts with \(\left[1 - \frac{(1-\theta)p}{1-(1-\theta)(1-p)}\right]^{q\hat{N}-1} \cdot \left(\frac{1-p}{p}\right)^{(q-1)\hat{N}-1} < \frac{1-\alpha}{\alpha}\). Therefore we have \(\hat{N} > \hat{N}^*\).

**Proof. (Theorem 3)** The reasoning for why O, II, and M strategies are not equilibrium strategies for strategic voters when \(\theta < \theta_{qp}\) and \(N > \hat{N}^*\) is the same as for the case in which \(\theta > \theta_{qp}\) and \(N > \hat{N}^*\), so the proof for this is omitted.

(3.1) Now I show that H strategy is not a Nash equilibrium strategy.

Consider a strategic voter who believes that all other strategic voters votes A regardless of their own private signals. Then in his belief,

\[
\beta_N^q = (1-\theta)(1-p) + \theta = 1 - (1-\theta)(1-p)
\]

and

\[
\beta_{WN}^q = (1-\theta)p + \theta = 1 - (1-\theta)p,
\]

thus

\[
\frac{1-\beta_{WN}^q}{1-\beta_N^q} = \frac{p}{1-p}.
\]

As \(N < \hat{N}\), i.e., \(N \leq \hat{N} - 1\), then by Corollary 3, we have

\[
B_N^q = \left[\frac{1-\alpha}{\alpha} \cdot \left(\frac{1-p}{p}\right)^{(1-q)N+1}\right]^{\frac{1}{qN-1}} < \frac{1-(1-\theta)p}{1-(1-\theta)(1-p)},
\]

thus
Then if this strategic voter receives signal $nw$, we have
\[ \frac{EU_N(A|nw,piv)}{EU_N(R|nw,piv)} = \frac{\alpha}{1-\alpha} \cdot \frac{p}{1-p} \cdot \left( \frac{\beta_{NW_N}^q}{\beta_{WN_N}^q} \right)^{qN-1} \cdot \left( \frac{1 - \beta_{NW_N}^q}{1 - \beta_{WN_N}^q} \right)^{(1-q)N} \]

which implies that $EU_N(A|nw,piv) < EU_N(R|nw,piv)$ as $EU_N(A|nw,piv) < 0$ and $EU_N(R|nw,piv) < 0$. Therefore this strategic voter’s best response when receiving signal $nw$ is to vote $R$, rather than votes $A$. Hence the H strategy is not a symmetric Nash equilibrium strategy for strategic voters.

Now we show that there exists a I strategy that constructs a symmetric Nash equilibrium, i.e., Type I Nash equilibrium exists.

Consider a strategic voter who believes that all other strategic voters are using the same I strategy $(\sigma_N^I(w), \sigma_N^I(nw))$ with $\sigma_N^I(w) = 1$ and $\sigma_N^I(nw) \in (0,1)$. Then this I strategy constructs a Nash equilibrium if and only if this strategic voter has
\[ EU_N(R|nw,piv) = EU_N(A|nw,piv) \]
and
\[ EU_N(R|w,piv) \leq EU_N(A|w,piv). \]

By Theorem 1 we know that once $EU_N(R|nw,piv) = EU_N(A|nw,piv)$, we must have $EU_N(R|w,piv) < EU_N(A|w,piv)$, so to show that the above I strategy is a Nash equilibrium strategy, we just need to show that it makes $EU_N(R|nw,piv)$ equal $EU_N(A|nw,piv)$.

As for this strategic voter,
\[ \beta_{NW_N}^q = (1 - \theta)(1 - p) + \theta \left[ (1 - p) + p\sigma_N^I(nw) \right] = 1 - \left[ 1 - \theta \sigma_N^I(nw) \right] p \]
and
\[ \beta_{WN_N}^q = (1 - \theta)p + \theta \left[ p + (1 - p)\sigma_N^I(nw) \right] = 1 - \left[ 1 - \theta \sigma_N^I(nw) \right] (1 - p), \]

then
\[ \frac{1 - \beta_{NW_N}^q}{1 - \beta_{WN_N}^q} = \frac{p}{1-p}, \]
by solving
\[ \frac{EU_N(R|nw,piv)}{EU_N(A|nw,piv)} = \frac{1 - \alpha}{\alpha} \cdot \frac{1 - p}{p} \cdot \left( \frac{\beta_{NW_N}^q}{\beta_{WN_N}^q} \right)^{qN-1} \cdot \left( \frac{1 - \beta_{NW_N}^q}{1 - \beta_{WN_N}^q} \right)^{(1-q)N} \]
\[ = \frac{1 - \alpha}{\alpha} \cdot \frac{1 - p}{p} \cdot \left( \frac{1 - \left[ 1 - \theta \sigma_N^I(nw) \right] (1 - p)}{1 - \left[ 1 - \theta \sigma_N^I(nw) \right] p} \right)^{qN-1} \cdot \left( \frac{1 - p}{p} \right)^{(1-q)N} \]
\[ = 1, \]
we have
\[ \sigma^q_N(nw) = \frac{1}{\theta} \cdot \frac{p B^q_N - (1-p)q}{p - (1-p)B^q_N} = \frac{1}{\theta} \cdot \sigma^q_{N^*}(nw), \]
where \( B^q_N = \left( \frac{1-\alpha}{\alpha} \right)^{\frac{1}{N-1}} \left( \frac{1-p}{p} \right)^{\frac{N}{N-1}}. \) As \( N < \hat{N}, \) we have \( \sigma^q_{N^*}(nw) < \theta \)
by Lemma 4, which implies that \( \sigma^q_N(nw) \in (0, 1). \)

From above we can see that the I strategy \((\sigma^q_N(w), \sigma^q_N(nw)) = (1, \frac{1}{\theta} \cdot \sigma^q_{N^*}(nw))\)
is a symmetric Nash equilibrium strategy, therefore Type I Nash equilibrium exists, and from above we can see that the Type I Nash equilibrium with above I strategy for strategic voters is the unique Type I Nash equilibrium when \( \theta < \theta_w \) and \( \hat{N} < N < \hat{N}. \)

(3.2) Since if \( N > \hat{N}, \) we must have \( N > \hat{N}^*. \) By the proof of (3.1), we can see that if Type I Nash equilibrium exists, the corresponding strategy of a strategic voter must be \((\sigma^q_N(w), \sigma^q_N(nw)) = (1, \frac{1}{\theta} \cdot \sigma^q_{N^*}). \) Yet if \( N > \hat{N}, \) we have \( \sigma^q_{N^*} > \theta \) and then \( \sigma^q_N(nw) = \frac{1}{\theta} \cdot \sigma^q_{N^*} > 1, \) which contradicts with \( \sigma^q_N(nw) \leq 1. \) Thus I strategy is not a Nash equilibrium strategy. So I can focus on showing that Type H Nash equilibrium exists as the unique symmetric Nash equilibrium.

Consider a strategic voter who believes that all other strategic voters adopt the H strategy, i.e., they vote A regardless of their signals. Then
\[ \beta^q_{NW,N} = (1-\theta)(1-p) + \theta = 1 - (1-\theta)p \]
and
\[ \beta^q_{NW,N} = (1-\theta)p + \theta = 1 - (1-\theta)(1-p). \]

As \( N > \hat{N}, B^q_N > \frac{1-(1-\theta)p}{1-(1-\theta)(1-p)}, \) then \( \frac{\beta^q_{NW,N}}{\beta^q_{NW,N}} < B^q_N, \) which implies that
\[
\frac{EU_N(A|nw, piv)}{EU_N(R|nw, piv)} = \frac{\alpha}{1-\alpha} \cdot \frac{p}{1-p} \cdot \left( \frac{\beta^q_{NW,N}}{\beta^q_{NW,N}} \right)^{qN-1} \cdot \left( \frac{1 - \beta^q_{NW,N}}{1 - \beta^q_{NW,N}} \right)^{(1-q)N}
\]
\[
= \frac{\alpha}{1-\alpha} \cdot \frac{p}{1-p} \cdot \left[ \frac{1 - (1-\theta)p}{1 - (1-\theta)(1-p)} \right]^{qN-1} \cdot \left( \frac{p}{1-p} \right)^{(1-q)N}
\]
\[
< \frac{\alpha}{1-\alpha} \cdot \frac{p}{1-p} \cdot \left( B^q_N \right)^{qN-1} \cdot \left( \frac{p}{1-p} \right)^{(1-q)N}
\]
\[
= \frac{\alpha}{1-\alpha} \cdot \frac{p}{1-p} \cdot \frac{1 - \alpha}{\alpha} \cdot \left( \frac{1-p}{p} \right)^{(1-q)N+1} \cdot \left( \frac{p}{1-p} \right)^{(1-q)N}
\]
\[
= 1.
\]

Then \( EU_N(A|nw, piv) > EU_N(R|nw, piv), \) which implies than when receiving signal nw, this strategic voter’s best response is to vote A. And by
Theorem 1, EU₅(A|nw, piv) > EU₅(R|nw, piv) implies EU₅(A|w, piv) > EU₅(R|w, piv), then his best response when receiving signal w is also to vote A. Therefore, H strategy is a Nash equilibrium strategy. Hence Type H Nash equilibrium exists, and it is the unique symmetric Nash equilibrium when N > ¹ and θ < θₚ.

Proof. (Proposition 3) By Theorem 2 we know that when N > ¹, the unique symmetric Nash equilibrium is the Type I Nash equilibrium in which each strategic voter adopts strategy (σₙ₅⁺(w), σₙ₅⁺(nw)) = (1, ¹ ⋅ σₙᵢ₅⁺(nw)). Then in this Nash equilibrium, we have

\[ \beta_N^q = (1 - \theta)p + \theta \left[ p + (1 - p)\sigma_N^{q^*}(nw) \right] \]

\[ = p + \theta(1 - p) \cdot \frac{1}{\theta} \cdot \sigma_N^{q^*}(nw) \]

\[ = p + (1 - p) \cdot \sigma_N^{q^*}(nw) \]

and

\[ \beta_N^{qNW} = (1 - \theta)(1 - p) + \theta \left[ (1 - p) + p\sigma_N^{q^*}(nw) \right] \]

\[ = 1 - p + \theta p \cdot \frac{1}{\theta} \cdot \sigma_N^{q^*}(nw) \]

\[ = 1 - p + p \cdot \sigma_N^{q^*}(nw). \]

As \( \sigma_N^{q^*}(nw) = \frac{pB_N^q - (1 - p)}{p - (1 - p)B_N^q} \) where \( B_N^q = \frac{(1 - q)}{\alpha} N^{-1} \cdot \left( \frac{1 - p}{p} \right)^{-1} \cdot (1 - \theta)^{-1} \), we can see that both \( \beta_N^q \) and \( \beta_N^{qNW} \) are independent of \( \theta \), which implies that the probabilities that mistakes happen,

\[ Pr_N(RE|W) = 1 - \sum_{m=q}^{N} \binom{N}{m} \left( \beta_N^q \right)^m \left( 1 - \beta_N^q \right)^{N-m} \]

and

\[ Pr_N(AP|NW) = \sum_{m=q}^{N} \binom{N}{m} \left( \beta_N^{qNW} \right)^m \left( 1 - \beta_N^{qNW} \right)^{N-m} \],

are independent of \( \theta \). ■

Recall that when \( q = q_{p\theta}, \left( \frac{1 - p}{p} \right)^{\frac{1 - q}{\theta}} = \frac{1 - (1 - \theta)p}{1 - (1 - \theta)(1 - p)} \), and \( q_{p\theta} \in (1 - (1 - \theta)p, 1 - (1 - \theta)(1 - p)) \).

Lemma 9 For any \( \{N, q, p, \theta, \alpha\} \) with \( N > ¹, q, p, \alpha \in \left( \frac{1}{2}, 1 \right) \) and \( \theta \in (0, 1) \),

(9.1) \( \sigma_N^{q^*}(nw) > \theta \) if and only if \( q > q_{p\theta} \);

(9.2) \( \sigma_N^{q^*}(nw) < \theta \) if and only if \( q < q_{p\theta} \).
Lemma 10 For any given \( \{q,p\} \) with \( q, p \in (\frac{1}{2}, 1) \) and \( \theta \in (0, 1) \),

(10.1) \( \theta < \theta_{q,p} \) implies \( q > q_{\theta,p} \), and vice versa;

(10.2) \( \theta > \theta_{q,p} \) implies \( q < q_{\theta,p} \), and vice versa.

Proof. (Proposition 4) As shown in Theorem 3, when \( N > \hat{N} \), Type II Nash equilibrium is the unique symmetric Nash equilibrium, in which

\[
\beta_{NW}^{q} = 1 - (1 - \theta)(1 - p)
\]

and

\[
\beta_{NW}^{q} = 1 - (1 - \theta)p.
\]

Then the probabilities that mistakes happen are

\[
Pr_N(\text{AP}|\text{NW}) = \sum_{m=qN}^{N} \binom{N}{m} \left( \beta_{NW}^{q} \right)^{m} \left( 1 - \beta_{NW}^{q} \right)^{N-m}
\]

\[
= \sum_{m=qN}^{N} \binom{N}{m} [1 - (1 - \theta)(1 - p)]^{m} [(1 - \theta)p]^{N-m}
\]

and

\[
Pr_N(\text{RE}|\text{W}) = 1 - \sum_{m=qN}^{N} \binom{N}{m} \left( \beta_{NW}^{q} \right)^{m} \left( 1 - \beta_{NW}^{q} \right)^{N-m}
\]

\[
= 1 - \sum_{m=qN}^{N} \binom{N}{m} [1 - (1 - \theta)(1 - p)]^{m} [(1 - \theta)(1 - p)]^{N-m}
\]

\[
= \sum_{m=0}^{qN-1} \binom{N}{m} [1 - (1 - \theta)(1 - p)]^{m} [(1 - \theta)(1 - p)]^{N-m}.
\]

Let’s check \( Pr_N(\text{AP}|\text{NW}) \) first. Notice that \( q > q_{\theta,p} \) implies the \( q > 1 - (1 - \theta)p \), as by Lemma 6, \( q_{\theta,p} > 1 - (1 - \theta)p \). Since

\[
\frac{\partial Pr_N(\text{AP}|\text{NW})}{\partial \theta}
\]

\[
= \sum_{m=qN}^{N} \binom{N}{m} p^{N-m} \left\{ mp[1 - (1 - \theta)p]^{m-1} (1 - \theta)^{N-m} - (N - m)[1 - (1 - \theta)p]^{m} (1 - \theta)^{N-m-1} \right\}
\]

\[
= \sum_{m=qN}^{N} \binom{N}{m} p^{N-m} (1 - p + \theta p)^{m-1} (1 - \theta)^{N-m-1} \{ [mp(1 - \theta) - (N - m)(1 - p + \theta p)] \}
\]

\[
= \sum_{m=qN}^{N} \binom{N}{m} p^{N-m} (1 - p + \theta p)^{m-1} (1 - \theta)^{N-m-1} (m + Np - Np\theta - N)
\]

\[
= \sum_{m=qN}^{N} \binom{N}{m} p^{N-m} (1 - p + \theta p)^{m-1} (1 - \theta)^{N-m-1} [m - (1 - p + p\theta)N],
\]

as \( \theta \in \Theta_q \) implies that \( q > q_{\theta,p} > 1 - (1 - \theta)p \), then for every \( m = qN, qN + 1, \ldots, N \) we have

\[
m - (1 - p + p\theta)N > [1 - (1 - \theta)p]N - (1 - p + p\theta)N = 0.
\]
which implies that
\[
\frac{\partial P_{\tau N}(AP|NW)}{\partial \theta} > 0,
\]
then \(P_{\tau N}(AP|NW)\) strictly increases with \(\theta\).

Now let’s turn to \(P_{\tau N}(RE|W)\). Since
\[
\frac{\partial P_{\tau N}(RE|W)}{\partial \theta} = \sum_{m=0}^{qN-1} \binom{N}{m} (1-p)^{N-m} \left\{ m(1-p)[1 - (1-\theta)(1-p)]^{m-1} (1-\theta)^{N-m} - (N-m)[1 - (1-\theta)(1-p)]^{m}(1-\theta)^{N-m} \right\}
\]
\[
= \sum_{m=0}^{qN-1} \binom{N}{m} (1-p)^{N-m} [m - (1-\theta)(1-p)]^{m-1} (1-\theta)^{N-m-1} [m(1-p)(1-\theta) - (N-m)(p+\theta-\theta p)]
\]
where \(m \leq qN - 1 < q_{\rho}N - 1 < [1 - (1-\theta)(1-p)]N - 1\), as by Lemma 6, \(q_{\rho} < 1 - (1-\theta)(1-p)\), we have
\[
\frac{\partial P_{\tau N}(RE|W)}{\partial \theta} < 0.
\]
Therefore, \(P_{\tau N}(RE|W)\) is strictly decreasing with \(\theta\). ■

Proof. (Proposition 5) Given \(N > \hat{N}\), the unique symmetric Nash equilibrium is the Type I one with \((\sigma^*_N(w), \sigma^*_N(nw))\) when \(\theta = \theta_1 \in \Theta_{\hat{N}}\), which implies that
\[
\beta^*_W |_{\theta=\theta_1} = p + (1-p)\sigma^*_N(nw) \text{ and } \beta^*_{NW} |_{\theta=\theta_1} = (1-p) + p\sigma^*_N(nw).
\]

On the other hand, when \(N > \hat{N}\) and \(\theta = \theta_2 \in \Theta_{\hat{N}}\), the unique symmetric Nash equilibrium is the Type H one with \((\sigma^*_H(w), \sigma^*_H(nw))\) when \(\theta = \theta_2 \in \Theta_{\hat{N}}\)
\[
\beta^*_W |_{\theta=\theta_2} = p + (1-p)\theta_2 \text{ and } \beta^*_H |_{\theta=\theta_2} = (1-p) + p\theta_2.
\]

By the definition of \(N > \hat{N}\) in Lemma 4, we have \(\sigma^*_N(nw) > \theta > \theta_2\), which implies \(\beta^*_W |_{\theta=\theta_1} > \beta^*_H |_{\theta=\theta_2}\) and \(\beta^*_W |_{\theta=\theta_1} > \beta^*_H |_{\theta=\theta_2}\) ■

Proof. (Lemma 6) Since \(p \in (\frac{1}{2}, 1)\) and \(\theta \in (0,1)\), we have \(\frac{1-(1-\theta)p}{1-(1-\theta)(1-p)} \in (\frac{1-p}{p}, 1)\).

Consider function \(\psi(q) = \left(\frac{1-p}{p}\right)^{1-\frac{q}{\theta}}\), which is continuous and strictly increasing with \(q\). Since \(\lim_{q\to\frac{1}{2}} \psi(q) = \frac{1-p}{p}\) and \(\lim_{q\to1} \psi(q) = 1\), these exist a unique \(q \in (\frac{1}{2}, 1)\), denoted as \(q_{\rho}\), such that \(\psi(q_{\rho}) = \frac{1-(1-\theta)p}{1-(1-\theta)(1-p)}\). Solving
which we have

\[ q_{p,q} = \frac{1}{\log_{\frac{1-\theta}{p}} \left[ \frac{1-(1-\theta)p}{1-(1-\theta)(1-p)} + 1 \right]}
\]

By the strict monotonicity of \( \psi(q) \) we have

\[ \left( \frac{1-p}{p} \right)^{\frac{1-q}{q}} = \psi(q) < \frac{1-(1-\theta)p}{1-(1-\theta)(1-p)}, \quad \forall q < q_{p,q}, \]

and

\[ \left( \frac{1-p}{p} \right)^{\frac{1-q}{q}} = \psi(q) > \frac{1-(1-\theta)p}{1-(1-\theta)(1-p)}, \quad \forall q > q_{p,q}. \]

As \( 0 < \frac{1-p}{p} < \frac{1-(1-\theta)p}{1-(1-\theta)(1-p)} < 1, \forall \theta \in (0,1) \), \( \ln \left( \frac{1-p}{p} \right) < \ln \left[ \frac{1-(1-\theta)p}{1-(1-\theta)(1-p)} \right] < 0 \),

which implies that \( q_{p,q} = \frac{\ln \left( \frac{1-p}{p} \right)}{\ln \left[ \frac{1-(1-\theta)p}{1-(1-\theta)(1-p)} \right]} \in \left( \frac{1}{2}, 1 \right) \).

Suppose \( q_{p,q} \geq 1 - (1-\theta)(1-p) \), as \( \ln \left( \frac{1-p}{p} \right) \) < 0 and \( \ln \left[ \frac{1-(1-\theta)p}{1-(1-\theta)(1-p)} \right] < 0 \),

we have

\[ \ln \left( \frac{1-p}{p} \right) \leq \left[ 1 - (1-\theta)(1-p) \right] \cdot \left\{ \ln \left( \frac{1-p}{p} \right) + \ln \left[ \frac{1-(1-\theta)p}{1-(1-\theta)(1-p)} \right] \right\} \]

\[ (1-\theta)(1-p) \cdot \ln \left( \frac{1-p}{p} \right) \leq \left[ 1 - (1-\theta)(1-p) \right] \cdot \ln \left[ \frac{1-(1-\theta)p}{1-(1-\theta)(1-p)} \right] \]

\[ (1-\theta)(1-p) \ln(1-p) - \ln p \leq \left[ 1 - (1-\theta)(1-p) \right] \cdot \left[ \ln(1-\theta) - \ln(1-p) \right] \]

\[ (1-\theta)(1-p) \cdot \ln p - \ln(1-p) \geq \left[ 1 - (1-\theta)(1-p) \right] \cdot \left[ \ln(1-\theta) - \ln(1-p) \right] \]

As \( f(x) = \ln x \) is continuous and differentiable on \([1-p,p]\) and \([1-(1-\theta)p, 1-(1-\theta)(1-p)]\), there exists \( x_1 \in (1-p, p) \) and \( x_2 \in (1-(1-\theta)p, 1-(1-\theta)(1-p)) \) such that

\[ \ln p - \ln(1-p) = (2p - 1) \cdot \frac{1}{x_1} \]

and

\[ \ln(1 - (1-\theta)(1-p)) - \ln(1 - (1-\theta)p) = (1-\theta)(2p - 1) \cdot \frac{1}{x_2}. \]

by which we have

\[ (1-\theta)(1-p)(2p - 1) \cdot \frac{1}{x_1} \geq \left[ 1 - (1-\theta)(1-p) \right] \cdot (1-\theta)(2p - 1) \cdot \frac{1}{x_2} \]

\[ \frac{x_2}{x_1} \geq \frac{1-(1-\theta)(1-p)}{1-p}. \]

Yet this contradicts with \( \frac{x_2}{x_1} < \frac{1-(1-\theta)(1-p)}{1-p} \), as \( x_1 > 1-p \) and \( x_2 < 1-(1-\theta)(1-p) \).

Therefore, \( q_{p,q} < 1 - (1-\theta)(1-p) \).

Similarly, to have \( q_{p,q} \leq 1 - (1-\theta)p \), we need to have \( y_i \) and \( y_j \) such that

\[ y_i \in (1-p, p), \quad y_j \in (1-(1-\theta)p, 1-(1-\theta)(1-p)), \]

and
\[
\frac{y_2}{y_1} \leq \frac{1 - (1 - \theta)p}{1 - (1 - \theta)(1 - p)}.
\]

Yet this contradicts with \( \frac{y_2}{y_1} > \frac{1 - (1 - \theta)p}{p} \), as \( y_1 < p \) and \( y_2 > 1 - (1 - \theta)p \), therefore we can show that \( q_{\theta} > 1 - (1 - \theta)p \).

**Proof. (Proposition 6)** By Lemma 10, \( q < q_{\theta} \) implies that \( \theta > \theta_{q_{\theta}} \), then the rest of the proof for this proposition is the same as the proof for Theorem 2.

**Proof. (Proposition 7)** By Lemma 10, \( q > q_{\theta} \) implies that \( \theta < \theta_{q_{\theta}} \), then the rest of the proof for this proposition is the same as the proof for Theorem 3.

**Lemma 11** Both \( \beta_{q_{WN}}^q \) and \( \beta_{q_{NW}}^q \) are strictly increasing with \( N \). Let

\[
\beta_{q_{WN}}^q \equiv \lim_{N \to \infty} \beta_{q_{WN}}^q = \frac{(2p - 1)p^{\frac{1}{2}} - 1}{p^{\frac{1}{2}} - (1 - p)^{\frac{1}{2}}},
\]

and

\[
\beta_{q_{NW}}^q = \lim_{N \to \infty} \beta_{q_{NW}}^q = \frac{(2p - 1)(1 - p)^{\frac{1}{2}} - 1}{p^{\frac{1}{2}} - (1 - p)^{\frac{1}{2}}},
\]

then \( \beta_{q_{WN}}^q > q > \beta_{q_{NW}}^q \).

**Lemma 12** Suppose \( t \in (0, 1) \). Define \( D_N(t) = \sum_{n=1}^{N} \binom{N}{m} t^m (1 - t)^{N - m} \), where \( q \in (0, 1) \) and \( qN \) is assumed to be an integer for any \( N \), then

(12.1) \( \lim_{N \to \infty} D_N(t) = 0 \) if \( t < q \);

(12.2) \( \lim_{N \to \infty} D_N(t) = 1 \) if \( t > q \).

**Proof.** Consider a sequence of i.i.d random variables \( X_1, X_2, ..., X_N \) such that \( Pr(X_n = 1) = t \) and \( Pr(X_n = 0) = 1 - t \), \( \forall n = 1, 2, ..., N \). Define \( \overline{X}_N = \frac{1}{N} \sum_{n=1}^{N} X_n \), then

\( D_N(t) = Pr(\overline{X}_N \geq q) \).

As \( E(X_n) = t \), \( \forall n = 1, 2, ..., N \), by the Law of Large Numbers, we have

\( \lim_{N \to \infty} Pr(|\overline{X}_N - t| < \epsilon) = 1, \forall \epsilon > 0 \).

(12.1) When \( t < q \), take any \( \epsilon \in (0, q - t) \). As \( Pr(|\overline{X}_N - t| \geq \epsilon) + Pr(|\overline{X}_N - t| < \epsilon) = 1 \), we have

\( \lim_{N \to \infty} Pr(|\overline{X}_N - t| \geq \epsilon) = 1 - \lim_{N \to \infty} Pr(|\overline{X}_N - t| < \epsilon) = 1 - 1 = 0. \)
Since $q > \epsilon + t$ implies that
\[ Pr(\overline{X}_N \geq q) < Pr(\overline{X}_N \geq \epsilon + t) < Pr(|\overline{X}_N - t| \geq \epsilon), \]
we have
\[ \lim_{N \to \infty} D_N(t) = \lim_{N \to \infty} Pr(\overline{X}_N \leq a) \leq \lim_{N \to \infty} Pr(|\overline{X}_N - t| \geq \epsilon) = 0, \]
while $Pr(\overline{X}_N \leq 0, \forall N$, implies that $\lim_{N \to \infty} D_N(t) \geq 0$. Therefore, $\lim_{N \to \infty} D_N(t) = 0$.

(12.2) When $t > q$, by
\[ Pr(\overline{X}_N \geq q) = Pr(\overline{X}_N - t \geq q - t) \geq Pr(t - q \geq \overline{X}_N - t \geq q - t) = Pr(\overline{X}_N - t \leq t - q) \geq Pr(\overline{X}_N - t < t - q) \]
we have
\[ \lim_{N \to \infty} Pr(\overline{X}_N \geq q) \geq \lim_{N \to \infty} Pr(|\overline{X}_N - t| < t - q) = 1, \]
while $\lim_{N \to \infty} Pr(\overline{X}_N \geq q) \leq 1$ as $Pr(\overline{X}_N \geq q) \leq 1$ for any $N$.

Therefore we know the conclusions hold.

Proposition 8 Consider any $\{N, q, p, \theta, \alpha\}$, where $q, p, \alpha \in (\frac{1}{2}, 1)$, $\theta \in (0, 1)$, and $q > q_{ps}$. Then when $N$ is large, i.e., $N > \hat{N}$, in the unique symmetric Nash equilibrium,

(8.1) if $q > 1 - (1 - \theta)(1 - p)$, information is not aggregated effectively, as
\[ \lim_{N \to \infty} Pr_N^q(\text{AP}|\text{NW}) = 0, \text{ but } \lim_{N \to \infty} Pr_N^q(\text{RE}|\text{W}) = 1; \]

(8.2) if $q < 1 - (1 - \theta)(1 - p)$, information is aggregated effectively, as
\[ \lim_{N \to \infty} Pr_N^q(\text{AP}|\text{NW}) = 0, \text{ and } \lim_{N \to \infty} Pr_N^q(\text{RE}|\text{W}) = 0. \]

Proof. As effective information aggregation is an asymptotic result, I just need to focus on large $N$. More precisely, I focus on Type H Nash equilibrium, as for any $N > \hat{N}$, it is the unique symmetric Nash equilibrium. The assumption that $q > q_{ps}$ guarantees that $\hat{N}$ is finite, so there exists $N > \hat{N}$.

In this Nash equilibrium, we have $\beta_{nwN}^q = 1 - (1 - \theta)p$ and $\beta_{wnN}^q = 1 - (1 - \theta)(1 - p)$. Then
\[
Pr(AP|NW) = \sum_{m=q}^{N} \binom{N}{m} \left( \beta_{NW}^q \right)^m \left( 1 - \beta_{NW}^q \right)^{N-m}
\]

and
\[
Pr(AP|W) = \sum_{m=q}^{N} \binom{N}{m} \left( \beta_{W}^q \right)^m \left( 1 - \beta_{W}^q \right)^{N-m}.
\]

For any \( q > q_{p\theta} \), by Lemma 6, we have \( \beta_{NW}^q = 1 - (1 - \theta)p < q_{p\theta} < q \). Then by Lemma 12 we have \( \lim_{N \to \infty} Pr(AP|NW) = 0 \).

(8.1) When \( q > 1 - (1 - \theta)(1 - p) \), we have \( \beta_{NW}^q < q \), then by Lemma 12 we have
\[
\lim_{N \to \infty} Pr(AP|W) = 0.
\]

Therefore, \( \lim_{N \to \infty} Pr(RE|W) = 1 - \lim_{N \to \infty} Pr(AP|W) = 1 \).

Since \( \lim_{N \to \infty} Pr_N^q(RE|W) = 1 \), that is, as the size of population of voters increases, though in the state in which in will work, the policy will be rejected by mistake all most sure. Thus information is not aggregated effectively.

(8.2) By Lemma 6, we know that \( \beta_{W}^q > q \), then by Lemma 12 we have
\[
\lim_{N \to \infty} Pr_N^q(AP|W) = 1,
\]

which implies that
\[
\lim_{N \to \infty} Pr(RE|W) = 1 - \lim_{N \to \infty} Pr_N^q(AP|W) = 0.
\]

Together with \( \lim_{N \to \infty} Pr_N^q(AP|NW) = 0 \), we know that information is aggregated effectively.

This completes the proof. ■

**Proposition 9** Consider any given \( \{q, p, \theta, \alpha\} \) with \( q, p, \alpha \in \left( \frac{1}{2}, 1 \right) \), \( \theta \in (0, 1) \), and \( q < q_{p\theta} \). Then information is aggregated effectively in the unique symmetric Nash equilibrium for large population. That is,
\[
\lim_{N \to \infty} Pr_N^q(AP|NW) = 0, \text{ and } \lim_{N \to \infty} Pr_N^q(RE|W) = 0.
\]

**Proof.** By Lemma 11 we know that there exist \( N_w \) and \( N_{NW} \) such that for any \( N > N_w \), \( \beta_{NW}^q > q \) and for any \( N > N_{NW} \), \( \beta_{NW}^q < q \). Take \( \bar{N} = \max\{N_w, N_{NW}, \hat{N}^*\} \), then for any \( N > \bar{N} \), we have

(9.1) Type I Nash equilibrium is the unique symmetric Nash equilibrium, in which
\[ \beta_{WN}^q = 1 - (1 - p) \left( 1 - \sigma_{N}^{*}(nw) \right) \] and \[ \beta_{NWN}^q = 1 - p \left( 1 - \sigma_{N}^{*}Nw \right) \],

where \( \sigma_{N}^{*}(nw) = \frac{pB_{N}^{q} - (1-p)B_{N}^{q}}{p - (1-p)B_{N}^{q}} \);

(9.2) \( \beta_{WN}^q > q \); 

(9.3) \( \beta_{NWN}^q < q \).

We cannot apply Lemma 12 directly here, as \( \beta_{WN}^q \) and \( \beta_{NWN}^q \) are not constants but vary with \( N \), which makes it kind difficult to characterize \( Pr_{N}^{q}(AP|W) \) and \( Pr_{N}^{q}(AP|NW) \) for finite \( N \). But they can still be characterized asymptotically.

Let’s look at \( Pr_{N}^{q}(AP|W) \) first. Define \( t_{W} = Pr_{N+1}^{q}(AP|W) \), then \( t_{W} > q \) is a constant. For any \( N > N \), we know that \( N \geq N_{+1} \), then \( \beta_{W}^q > \beta_{WN}^q \geq t_{W} > q \).

Define a sequence of i.i.d random variables \( X_{N_n} \), \( X_{N_2} \), ..., \( X_{N_N} \) such that

\[ E(X_{N_n}) = \beta_{WN}^q \] and \[ Var(X_{N_n}) = \beta_{WN}^q \left( 1 - \beta_{WN}^q \right) \]. Let \( \overline{X}_{N_N} = \sum_{n=1}^{N} X_{N_n} \), then

\[ Pr_{N}^{q}(AP|W) = \sum_{m=q}^{N} \binom{N}{m} \left( \beta_{NWN}^q \right)^m \left( 1 - \beta_{NWN}^q \right)^{N-m} \] 

\[ = Pr \left( \sum_{n=1}^{N} X_{N_n} \geq qN \right) \] 

\[ = Pr(\overline{X}_{N_N} \geq q) \]

By Chebyshev’s inequality, for any \( \epsilon > 0 \), we have

\[ Pr \left( |\overline{X}_{N_N} - \beta_{WN}^q| \geq \epsilon \right) \leq \frac{\beta_{WN}^q \left( 1 - \beta_{WN}^q \right)}{N \epsilon^2} \]

then \( Pr \left( |\overline{X}_{N_N} - \beta_{WN}^q| < \epsilon \right) \geq 1 - \frac{\beta_{WN}^q \left( 1 - \beta_{WN}^q \right)}{N \epsilon^2} \).

As \( q < \beta_{WN}^q < \beta_{W}^q \), we have

\[ Pr(\overline{X}_{N_N} \geq q) = Pr(\overline{X}_{N_N} - \beta_{WN}^q \geq q - \beta_{WN}^q) \] 

\[ \geq Pr(\beta_{WN}^q - q \geq \overline{X}_{N_N} - \beta_{WN}^q \geq q - \beta_{WN}^q) \] 

\[ = Pr(\overline{X}_{N_N} - \beta_{WN}^q \leq \beta_{WN}^q - q) \] 

\[ \geq Pr(\overline{X}_{N_N} - \beta_{WN}^q \leq t_{W} - q) \] 

\[ \geq 1 - \frac{\beta_{WN}^q \left( 1 - \beta_{WN}^q \right)}{N(t_{W} - q)^2} \],

since \( \beta_{WN}^q \geq t_{W} \).
As $Pr(\overline{X}_{NN} \geq q) < 1$, \( \forall N \), by
\[
1 > \lim_{N \to \infty} Pr(\overline{X}_{NN} \geq q) \\
\geq \lim_{N \to \infty} \left[ 1 - \frac{\beta_{qW_N} (1 - \beta_{qW_N}^q)}{N(t_W - q)^2} \right] \\
= 1 - \frac{1}{(t_W - q)^2} \cdot \lim_{N \to \infty} \frac{\beta_{qW_N}^q (1 - \beta_{qW_N}^q)}{N} \\
= 1,
\]
then $\lim_{N \to \infty} Pr^q_N (AP|W) = \lim_{N \to \infty} Pr(\overline{X}_{NN} \geq q) = 1$, and therefore $\lim_{N \to \infty} Pr^q_N (RE|W) = 1 - \lim_{N \to \infty} Pr^q_N (AP|W) = 0$.

Now let’s turn to $Pr^q_N (AP|NW)$. For any given $N$ and $m$ such that $m \geq qN$, the function $\zeta_m(x) = x^m (1-x)^{N-m}$ is strictly increasing on $(0,q)$, as
\[
\zeta_m'(x) = mx^{m-1} (1-x)^{N-m-1} (m-Nx) \\
\geq mx^{m-1} (1-x)^{N-m-1} (qN-Nx) \\
= mx^{m-1} (1-x)^{N-m-1} N(q-x) \\
> 0
\]
for any $x \in (0,q)$, which implies that
\[
Pr^q_N (AP|NW) = \sum_{m=qN}^{N} \binom{N}{m} \left( \beta_{qW_N}^q \right)^m \left( 1 - \beta_{qW_N}^q \right)^{N-m} \\
= \sum_{m=qN}^{N} \binom{N}{m} \zeta_m \left( \beta_{qW_N}^q \right) \\
\leq \sum_{m=qN}^{N} \binom{N}{m} \zeta_m \left( \beta_{qW}^q \right) \\
= \sum_{m=qN}^{N} \binom{N}{m} \left( \beta_{qW}^q \right)^m \left( 1 - \beta_{qW}^q \right)^{N-m}.
\]
Then if $\lim_{N \to \infty} Pr^q_N (AP|NW)$ exists, we must have
\[
\lim_{N \to \infty} Pr^q_N (AP|NW) \leq \lim_{N \to \infty} \sum_{m=qN}^{N} \binom{N}{m} \left( \beta_{qW}^q \right)^m \left( 1 - \beta_{qW}^q \right)^{N-m}.
\]
Also we must have $\lim_{N \to \infty} Pr^q_N (AP|NW) \geq 0$, as $Pr^q_N (AP|NW) \geq 0$, for any $N$. By Lemma 12 we know that $\lim_{N \to \infty} \sum_{m=qN}^{N} \binom{N}{m} \left( \beta_{qW}^q \right)^m \left( 1 - \beta_{qW}^q \right)^{N-m} = 0$ as $\beta_{qW}^q < q$. Therefore,
\[
0 \leq \lim_{N \to \infty} Pr^q_N (AP|NW) \leq 0,
\]
which implies that $\lim_{N \to \infty} Pr^q_N (AP|NW)$ exists and equals 0.

As $\lim_{N \to \infty} Pr^q_N (RE|W) = 0$ and $\lim_{N \to \infty} Pr(AN|NW) = 0$, we know that the probabilities that mistakes happen, $Pr^q_N (RE|W)$ and $Pr^q_N (AP|NW)$, diminish to arbitrarily small when the population of voters increases in the unique
symmetric Nash equilibrium. Therefore information is aggregated effectively. ■

**Proof. (Theorem 5)** Proof is straightforward from Proposition 8 and Proposition 9. ■

**Proof. (Theorem 4)** By Lemma 10 we know that Proposition 8 holds for \( \theta > \theta_{qp} \), while Proposition 9 holds for \( \theta < \theta_{qp} \), given any \( q \in \left( \frac{1}{2}, 1 \right) \), thus the result is straightforward from these two propositions. ■

**Proof. (Theorem 6)** Since in both cases of (6.1) and (6.2), voting \( A \) always maximizes strategic voters’ expected utility, regardless what signals they receive, to see how strategic heterogeneity affect strategic voters’ expected utilities, I can just look at the influence of \( \theta \) on \( EU(A|s, piv) \), a strategic voter’s expected utility of voting \( A \) when he receives signal \( s \in \{w, nw\} \).

Consider a strategic voter who receives signal \( s \in \{w, nw\} \), when the total population of voters is \( N \). His utility from voting \( A \) is:

\[
EU_N(A|s, piv) = -\alpha \cdot Pr_N(NW|s, piv) \\
= -\alpha \cdot Pr_N(NW) \cdot Pr_N(piv|NW) \cdot Pr_N(s|NW, piv) \\
= -\alpha \cdot Pr(s|NW) \cdot Pr_N(piv|NW) \\
= -2\alpha \cdot Pr(s|NW) \cdot \frac{1}{1 + \frac{Pr_N(piv|NW)}{Pr_N}\left(\frac{Pr_N(piv|NW)}{Pr_N}\right)} \\
\]

which implies that \( EU_N(A|s, piv) \) changes in the same direction as how \( \frac{Pr_N(piv|NW)}{Pr_N}\left(\frac{Pr_N(piv|NW)}{Pr_N}\right) \) changes in the same direction as how \( \frac{Pr_N(piv|NW)}{Pr_N}\left(\frac{Pr_N(piv|NW)}{Pr_N}\right) \) changes. Then I just need to check the influence of \( \theta \) on \( \frac{Pr_N(piv|NW)}{Pr_N}\left(\frac{Pr_N(piv|NW)}{Pr_N}\right) \).

Recall that \( Pr_N(piv|S) = \left(\frac{N}{q_{N-1}}\right) \cdot \left(\beta_N \right)^{q_{N-1}} \cdot (1-\beta_N)^{(1-q)N} \), \( S \in \{W, NW\} \).

(6.1) If \( \theta \in \Theta_{q+} \) and \( N > \hat{N} \), or \( \theta \in \Theta_{q-} \) and \( \hat{N} < N < \hat{N} \), in the unique Nash equilibrium, strategic voters adopt the mixed strategy profile \((\sigma_N^q(w), \sigma_N^q(nw)) = (1, \frac{1}{\theta} \cdot \sigma_N^{pF}(nw))\), then

\[
\beta_N^q = (1-\theta) \cdot Pr(w|S) + \theta \cdot \left[ Pr(w|S) \cdot \sigma_N^q(w) + Pr(nw|S) \cdot \sigma_N^q(nw) \right] \\
= (1-\theta) \cdot Pr(w|S) + \theta \cdot \left[ Pr(w|S) + Pr(nw|S) \cdot \frac{1}{\theta} \cdot \sigma_N^{pF}(nw) \right] \\
= Pr(w|S) + Pr(nw|S) \cdot \sigma_N^{pF}(nw) \\
\]

for \( S \in \{W, NW\} \), which implies that \( \frac{Pr_N(piv|W)}{Pr_N}\left(\frac{Pr_N(piv|NW)}{Pr_N}\right) \) is independent of \( \theta \), as \( Pr(w|S) \) and \( Pr(nw|S) \) are exogenous, and \( \sigma_N^{pF}(nw) \) is irrelevant with \( \theta \). Therefore, \( EU_N (A|s, piv) \) is independent of \( \theta \).

(6.2) Given \( \theta \in \Theta_{q-} \) and \( N > \hat{N} \), in the unique Nash equilibrium, strategic
voters adopt the Type-H strategy profile, then
\[
\begin{align*}
\beta^q_{NW} &= (1 - \theta) \cdot \Pr(w|W) + \theta = p + (1 - p) \cdot \theta \\
\beta^q_{NW} &= (1 - \theta) \cdot (1 - p) + \theta = 1 - p + \theta p.
\end{align*}
\]

As
\[
\frac{\Pr_N(piv|W)}{\Pr_N(piv|NW)} = \frac{\beta^q_{NW}}{\beta^q_{SW}} = \left( \frac{1 - \beta^q_{NW}}{1 - \beta^q_{SW}} \right)^{qN-1} q^nN \cdot \left( \frac{1 - \beta^q_{NW}}{1 - \beta^q_{SW}} \right)^{(1-q)N}
\]

is strictly decreasing with \( \theta \), we know that \( EU_N(A|s,piv) \) also decreases with \( \theta \).

\[ \text{Proof. (Theorem 7)} \] By the proof of Theorem 6, we know that we just need to check how \( q \) affects \( \frac{\Pr_N(piv|W)}{\Pr_N(piv|NW)} \), as strategic voters’ expected utility equals \( EU_N(A|s,piv) \) in the unique Nash equilibrium when the population is large, and both \( EU_N(A|s,piv) \) and \( \frac{\Pr_N(piv|W)}{\Pr_N(piv|NW)} \) changes in the same direction when \( q \) changes.

(7.1) When \( q \in Q_{un} \) and \( N > N^* \), or \( q \in Q_{pas} \) and \( N^* < N < \hat{N} \), in the unique symmetric Nash equilibrium, strategic voters adopt the Type-I strategy profile \( (\sigma^q_N(w), \sigma^q_N(nw)) = (1, \frac{1}{\theta} \cdot \sigma^{qF*}(nw)) \), then
\[
\begin{align*}
\beta^q_{NW} &= p + (1 - p) \cdot \sigma^{qF*}(nw) \\
\beta^q_{NW} &= 1 - p + p \cdot \sigma^{qF*}(nw)
\end{align*}
\]
and
\[
\frac{\Pr_N(piv|W)}{\Pr_N(piv|NW)} = \left[ \frac{p + (1 - p) \cdot \sigma^{qF*}(nw)}{1 - p + p \cdot \sigma^{qF*}(nw)} \right]^{qN-1} q^nN \cdot \left( \frac{1 - p}{p} \right)^{(1-q)N}.
\]

By \( \sigma^{qF*}(nw) = \frac{pB_q^q - (1-p)}{p -(1-p)B_q^q} \), we have
\[
\frac{\Pr_N(piv|W)}{\Pr_N(piv|NW)} = \left( \frac{1 - p}{p} \right)^{qN-1} q^nN \cdot \left( \frac{1}{B_q^q} \cdot \frac{p}{1-p} \right)^{(1-q)N},
\]
then \( \frac{\Pr_N(piv|W)}{\Pr_N(piv|NW)} \) and \( \ln \left( \frac{1}{B_q^q} \cdot \frac{p}{1-p} \right)^{qN-1} \) change in the same direction when \( q \) changes.

Notice that
\[
\frac{\partial B_q^q}{\partial q} = B_q^q \cdot \frac{N}{qN - 1} \cdot \ln \left( \frac{1}{B_q^q} \cdot \frac{p}{1-p} \right)^{qN-1},
\]

since \( B_q^q = \left( \frac{1 - \alpha}{\alpha} \right)^{qN-1} \cdot (1 - \frac{p}{1-p})^{qN-1} \), then
\[
\partial \left[ \ln \left( \frac{1}{B_q^q} \cdot \frac{p}{1-p} \right)^{qN-1} \right] = N \cdot \ln \left( \frac{1}{B_q^q} \cdot \frac{p}{1-p} \right) - (qN - 1) \cdot \frac{1}{B_q^q} \cdot \frac{\partial B_q^q}{\partial q} = 0.
\]
Therefore \( \ln \left( \frac{1}{B_N} \cdot \frac{p}{1-p} \right)^{qN-1} \) does not change with \( q \), which implies that \( EU_N(A|s, piv) \) is independent of \( q \).

When \( q \in Q_{\theta^+} \) and \( N \bar{N} \), in the unique symmetric Nash equilibrium, strategic voters adopt the Type-H strategy profile and then

\[
\begin{align*}
\beta_{NW_N}^q &= (1 - \theta) \cdot \Pr(w|W) + \theta = p + (1 - p) \cdot \theta \\
\beta_{NWN_N}^q &= (1 - \theta) \cdot (1 - p) + \theta = 1 - p + \theta p.
\end{align*}
\]

As

\[
\frac{\Pr_N(piv|W)}{\Pr_N(piv|NW)} = \left( \frac{\beta_{NW_N}^q}{\beta_{NWN_N}^q} \right)^{qN-1} \cdot \left( \frac{1 - \beta_{NW_N}^q}{1 - \beta_{NWN_N}^q} \right)^{(1-q)N}
\]

is strictly increasing with \( q \), we have \( EU(A|s, piv) \) strictly increase with \( q \).
Acknowledgement

I thank John Nachbar, Elizabeth Penn, and Marcus Berliant for their helpful guidance and encouragement. I also thank Scott Baker, Randall Calvert, John Patty, Werner Ploberger, Maher Said, Yunfei Cao, Wei-Cheng Chen, and Bo Li for their invaluable comments.

Reference


